

## ON LOCAL AND GLOBAL CONJUGACY

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## 1. INTRODUCTION

Let  $G$  be a reductive linear algebraic group over  $\mathbb{C}$  and  $H$  a closed subgroup of  $G$ , one may ask to what extent from the representation theory we can determine  $H$ . For example, the dimension data for  $H$  in  $G$  consists of integers  $m_H(\rho)$  where  $\rho$  runs through all finite dimensional representations of  $G$ , where  $m_H(\rho)$  is the multiplicity of the trivial representation in the restriction of  $\rho$  to  $H$ . R. Langlands ([13], [14]) asked, is the dimension data determines the isomorphism class and the conjugacy class of  $H$ . In this note, we discuss and classify a special family of counter examples (called LFMO-special representation, will be defined explicitly later) for  $G = \mathrm{SO}(2N)$  and  $H$  connected reductive, and give negative answer to Langlands' question in some sense. In fact, our first family of such examples (Theorem 3.15, also see [28] and [29]) gave first connected instances of locally conjugate subgroups of  $G = \mathrm{SO}(2N)$  failing to be conjugate. Moreover, with sufficient functoriality, such counter examples will give failure of multiplicity one, and thus got some attention in the study of beyond endoscopy which is much hotter after the establishment of the fundamental lemma by NGO ([9]).

In 1990, M. Larsen and R. Pink studied the case  $G = \mathrm{GL}(n)$ , and got some results on dimension data ([18]). In fact, they proved, for general  $G$  and connected reductive  $H$ , the isomorphism class of  $H$  is determined by the dimension data, and when  $G = \mathrm{GL}(n)$  and  $H$  embeds into  $G$  in an essential way, i.e., irreducible

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as  $n$ -dimensional representation, the conjugacy class of  $H$  is determined by the dimension data.<sup>1</sup> For general  $G$ , this is not the case. In fact, earlier works exhibited various counter examples ([4], [16], [17], [28], [30], [31], [29]).

The discrepancy between the representation feature and the conjugacy occurs due to various reason, and one is the “local-global issue”. Let  $G$  as above and  $H, H'$  two closed subgroups of  $G$ . We say that  $H$  and  $H'$  be *locally conjugate* or *element-wise conjugate* if there is an isomorphism  $i : H \xrightarrow{\cong} H'$  such that for  $h \in H$  in a (Zariski) dense subset of  $H$ ,  $h$  and  $i(h)$  are conjugate. We say that  $H$  and  $H'$  are *globally conjugate* if they are conjugate in  $G$ . Also, there are definitions in term of group homomorphisms. Let  $\rho, \rho' : H \rightarrow G$  be two homomorphisms of linear algebraic groups. We say that  $\rho$  and  $\rho'$  are *locally conjugate* if for  $h \in H$  in a (Zariski) dense subset of  $H$   $\rho(h)$  and  $\rho'(h)$  are conjugate in  $G$ . We say that  $\rho$  and  $\rho'$  are *globally conjugate* if they are conjugate, namely, there exists a  $g \in G$  such that  $\rho'(h) = g\rho(h)g^{-1}$  for all  $h$  in  $H$ . We say that  $\rho$  and  $\rho'$  are *globally conjugate in image* if  $\rho(H)$  and  $\rho'(H)$  are conjugate in  $G$ .

Of course, global conjugacy implies local conjugacy. Moreover there are subtle difference in definitions between the subgroup version and the group homomorphism version. A lot of representation features (e.g. dimension data) are closely related to the local conjugacy. So the question at the beginning is closely related to the difference between the local and global conjugacy issue.  $GL(n)$  case is simple since by the character theory, local conjugacy implies global conjugacy. For general  $G$ , this is not the case. When  $H$  is finite, it is much easier to find the example of local conjugacy without global conjugacy ([16], [17]). When  $H$  is connected, some pure (but not so trivial) representation reasons will tell. For more discussion, see Section 2.

We are interested in such question for several reasons. One is related to the multiplicity. When  $G = GL(n)$ , local conjugacy implies global conjugacy, and then this reflects the famous multiplicity one property for  $GL(n)$ . For  $G = PGL(n)$ , local conjugate subgroups might not be conjugate, and this reflects the failure of multiplicity one property in  $SL(n)$  ([4], [15]). When  $G = SO(2N)$ , first connected instance of local conjugate but not globally conjugate subgroups were found, and this induces an example the failure of multiplicity for  $SO(2N)$  with some assumptions on functoriality ([28]), and such example is totally different the ones before and hence answered Langlands’ question in some sense ([28], [9]). In fact, when  $G = SO(10)$ ,  $H = SO(5)$ ,  $\rho = \Lambda^2 : H \rightarrow G$  the exterior square representation, and  $\rho' = \tau \circ \rho$  where  $\tau$  be an outer automorphism of  $SO(10)$  (i.e., a conjugation by an element of  $O(10)$  with determinant  $-1$ ),  $\rho$  and  $\rho'$  are locally conjugate but not globally conjugate in image. Assuming sufficient functoriality, when starting from a tempered modular 4-dim  $l$ -adic Galois representation of full  $GSp(4)$  type which is easily obtained from the Tate module of a principally polarized generic Abelian surface, we get a 10-dim orthogonal  $l$ -adic Galois representation, which is associated to a (stable) cusp form of  $SO(10)$  of multiplicity  $> 1$ . in fact, we found several families of such example. The later work is to remove assumptions as much as we can.

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<sup>1</sup>But for general  $H$  in  $GL(n)$ , the conjugacy of  $H$  is not completely determined by the dimension data. The examples in [18] is not correct, and the correct example is shown in [30].

According our philosophy, local conjugacy without global conjugacy in image will leads to the failure of multiplicity one with sufficient functoriality condition. For local conjugacy VS global conjugacy in group homomorphism form, there are some subtle things, and it seems more strong assumption is needed to get the failure of multiplicity one.

This paper is to gather most of our local-global results in purely a representation way. For discussions related to automorphic form, see [29]. Throughout, unless specified, a homomorphism means a group homomorphism of complex algebraic groups, and a representation means a finite dimensional complex representation.

**Theorem A.** (a) Let  $G = \mathrm{GL}(N, \mathbb{C}), \mathrm{SL}(N, \mathbb{C}), \mathrm{Sp}(2N, \mathbb{C}), \mathrm{SO}(2N + 1, \mathbb{C})$  or  $\mathrm{O}(N, \mathbb{C})$ , or the group isogenous to above (except for  $\mathrm{O}(2N, \mathbb{C})$ ),  $H$  a connected complex reductive group,  $\rho, \rho': H \rightarrow G$  be two homomorphisms. If  $\rho$  and  $\rho'$  are locally conjugate, then they are globally conjugate.

(b) Let  $G = \mathrm{SO}(2N, \mathbb{C})$  or its isogenous form, and  $H, \rho, \rho'$  as (a). If  $\rho$  and  $\rho'$  are locally conjugate, then they either are globally conjugate, or differ by an outer automorphism.

*Remark:* Two connected algebraic groups  $G, G'$  are said to be *isogenous*, if there is a connected algebraic group  $G''$  together with two finite central isogenies  $G'' \rightarrow G$  and  $G'' \rightarrow G'$ . We also say that  $G'$  is an *isogenous form* of  $G$ . For explicit definitions, see Section 2.3.

This result for  $G$  classical is well known in representation theory, at least at the time of Dynkin ([7], [8]). For completeness we include all proof in this paper.

When  $G = \mathrm{SO}(2N, \mathbb{C})$ , or its isogenous form, if  $\rho, \rho': H \rightarrow G$  are locally conjugate, then they must either be globally conjugate or differ by an outer automorphism. So in such case, we need just study the case  $\rho$  and  $\rho' = \tau \circ \rho$  where  $\tau$  is an outer automorphism of  $G$  which is induced by a conjugation of an element of  $\mathrm{O}(2N)$  of determinant  $-1$ . Moreover, explicit analysis on multiplicities enable us to just focus on the case  $G = \mathrm{SO}(2N)$ .

**Theorem B.** Let  $G = \mathrm{SO}(2N, \mathbb{C})$ ,  $H$  a connected reductive group,  $\rho: H \rightarrow G$  be a homomorphism, and  $\rho' = \tau \circ \rho$  where  $\tau$  be an outer automorphism of  $G$ . Then we have the following:

(a)  $\rho$  and  $\rho'$  are locally conjugate if and only if  $\rho$  has weight 1, namely, the restriction of  $\rho$  to the maximal torus has trivial representation as its subrepresentation.

(b)  $\rho$  and  $\rho'$  are globally conjugate if and only if  $\rho$  has an odd dimensional orthogonal subrepresentation.

(c) Assume moreover that  $\rho$  is injective.  $\rho$  and  $\rho'$  are globally conjugate in image if and only if some automorphism of  $H$  lifts to an outer automorphism of  $\mathrm{SO}(2N)$ , i.e., a conjugation by an element of  $\mathrm{O}(2N)$  of determinant  $-1$ .

We say that a homomorphism  $\rho: H \rightarrow \mathrm{SO}(2N, \mathbb{C})$  is *LFMO-special*, if, for  $\rho' = \tau \circ \rho$  for an outer automorphism  $\tau$  of  $\mathrm{SO}(2N)$ , the following conditions for  $\rho$  and  $\rho'$  hold.

(1)  $\rho$  is *essential*, namely, the image of  $\rho$  is not contained in any parabolic subgroup of  $\mathrm{SO}(2N)$ . Or equivalently, as a representation  $\rho$  has no nontrivial totally isotropic subrepresentation.

(2)  $\rho$  and  $\rho'$  are locally conjugate.

(3)  $\rho$  and  $\rho'$  are not globally conjugate in image.

In this case, we also say that the  $2N$ -dimensional representation induced by  $\rho$  is LFMO-special.

If  $\rho$  is LFMO-special, then  $\rho$  and  $\rho'$  will be “leading to failure of multiplicity one”. That is also why we call this name. For reasons, see Theorem 3.2 of [29].

**Theorem C.** *Let  $H$  be a complex connected reductive group and  $\rho : H \rightarrow G = \mathrm{SO}(2N, \mathbb{C})$  a homomorphism. Then  $\rho$  is LFMO-special if and only if:*

(1) *The representation space of  $\rho$  decomposes as a direct sum of inequivalent even dimensional orthogonal subrepresentations.*

(2)  *$\rho$  has weight 1.*

(3) *If an automorphism  $\phi$  of  $\rho(H)$  lifts to an automorphism of  $G$ , then it lifts to an inner automorphism of  $G$ .*

*Moreover, if  $\rho'$  is quasi-equivalent to the representation induced by  $\rho$ , then  $\rho'$  is LFMO-special if and only if so is  $\rho$ .*

We say that  $\rho$  is *stable* if the  $2N$ -representation induced by  $\rho$  (also called  $\rho$ ) is irreducible. Let  $H = T \times H_1 \times \dots \times H_r$ , where  $T$  is a complex torus,  $H_i$  are simple Lie groups and  $\rho$  is a representation of  $H$ . Then by basic representation theory, we have  $\rho = 1 \otimes \bigotimes_i \rho_i$  where  $\rho_i$  is a representation of  $H_i$ , either symplectic or orthogonal. For general connected  $H'$ ,  $\rho'$  must be quasi-equivalent to some  $\rho$  of  $H$  of the above case. Moreover, when  $\rho$  is a stable LFMO-special representation,  $\rho'$  must have weight 1, and hence  $\rho'$  must kill the center of  $H'$ . Thus  $\rho'$  is quasi-equivalent to  $\rho$  of  $H$  for some  $H$  is semisimple of adjoint type. So the study of stable LFMO-special representation can be reduced to the case when  $H$  is semisimple of adjoint type.

**Theorem D.** *Let  $H = T \times H_1 \times \dots \times H_r$ ,  $\rho = \bigotimes_i \rho_i$  be an even dimensional orthogonal representation of a complex connected reductive group where  $T$  is a complex torus,  $H_i$  is a simple Lie group of adjoint type, and  $\rho_i$  is self-dual and irreducible for each  $i$ .*

*Then  $\rho$  is LFMO-special if and only if  $\rho(H)$  has trivial center, and one of the following cases happens:*

Case (1): *Exactly one  $\rho_i$  is even dimensional. In this case such  $\rho_i$  is orthogonal and LFMO-special.*

Case (2): *Exactly two, say  $\rho_i$  and  $\rho_j$ , are even dimensional. In this case the following must be excluded:  $H_i$  and  $H_j$  are isogenous, and  $\rho_i$  and  $\rho_j$  are quasi-equivalent, and moreover the dimension of  $\rho_i$  is  $\equiv 2 \pmod{4}$ .*

Case (3): *At least three  $\rho_i$ s are even dimensional.*

*Remark:* We say that two representations  $\rho$  and  $\rho'$  of  $H$  and  $H'$  are *quasi-equivalent* if and only if there are finite central isogenies  $\iota : H'' \rightarrow H$  and  $\iota' : H'' \rightarrow H'$  such that as representations of  $H''$ ,  $\rho \circ \iota$  and  $\rho' \circ \iota'$  are conjugate.

In particular, we have the following corollary, which shows up also in [28] and [29].

**Corollary E.** *Let  $H = H_1 \times H_2 \times \dots \times H_r$  be a semisimple Lie group of adjoint type over  $\mathbb{C}$  with simple factors  $H_j$  ( $1 \leq j \leq r$ ), among which at least one has even rank. Let  $V = \mathfrak{h}_1 \otimes \mathfrak{h}_2 \otimes \dots \otimes \mathfrak{h}_r$ ,  $\kappa_i$  the Killing form of  $\mathfrak{h}_i = \text{Lie}(H_i)$ , and  $\kappa = \kappa_1 \otimes \kappa_2 \otimes \dots \otimes \kappa_r$ .*

$$\rho = \text{Ad}_{H_1} \otimes \text{Ad}_{H_2} \otimes \dots \otimes \text{Ad}_{H_r} : H \rightarrow G = \text{SO}(V, \kappa) \cong \text{SO}(2N, \mathbb{C})$$

*Then  $\rho$  is LFMO-special if and only if one of the following happens:*

- (a) *Only one of  $H_j$  is of even rank, and  $\mathfrak{h}_j$  is  $A_{4n}(n \geq 1), B_{2n}(n \geq 1), C_{2n}(n \geq 1), E_6, E_8, F_4, G_2$ .*
- (b) *Exactly two of them, say  $H_j$  and  $H_k$ , are of even rank, and either  $H_j \not\cong H_k$ , or  $H_j \cong H_k$  with  $4 \mid \dim H_j$ .*
- (c) *At least three of them are of even rank.*

*In this case  $\rho$  will lead to the failure of multiplicity one for  $\text{SO}(2N)$ .*

*Remark:* For condition (b),  $4 \mid \dim H_j$  if and only if  $\mathfrak{h}_j$  is of type  $A_{2n}, B_{4n}, C_{4n}, D_{4n}, E_8, F_4$ , as

$$\begin{array}{lll} \dim A_{2n} = 4n(n+1), & \dim B_{2n} = 2n(4n+1) & \dim C_{2n} = 2n(4n+1), \\ \dim D_{2n} = 2n(4n-1), & \dim E_6 = 78 & \dim E_8 = 248 \\ \dim F_4 = 52 & \dim G_2 = 14 & \end{array}$$

The article is organized as following. In Section 2, we will focus on the structure theory of self-dual representations of connected reductive groups, and prove Theorem A. In Section 3, we will study the  $\text{SO}(2N)$  case, and finally prove all other theorems. Some parts are listed for the purpose of clarification of notations and concepts, and experts can skip them.

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## 2. STRUCTURE THEORY

### 2.1. Preliminaries.

First let's have some notations and preliminary discussions. Let  $G$  and  $H$  be complex reductive groups. Recall that two homomorphisms  $\rho$  and  $\rho'$  are *locally conjugate* if for  $h \in H$  in a (Zariski) dense open subset in  $H$ ,  $\rho(h)$  and  $\rho'(h)$  are conjugate in  $G$ ;  $\rho$  and  $\rho'$  are *globally conjugate* if  $\rho$  and  $\rho'$  differ by a conjugation by an element in  $G$ , i.e., there exists a  $g \in G$  such that  $\rho'(x) = g\rho(x)g^{-1}$  for all

$x \in H$ ; And moreover,  $\rho$  and  $\rho'$  are *globally conjugate in image* if  $\rho(H)$  and  $\rho'(H)$  are conjugate in  $G$ , i.e., there is a  $g \in G$  such that  $\rho'(H) = g\rho(H)g^{-1}$ .

Of course if  $\rho$  and  $\rho'$  are globally conjugate, then they are locally conjugate. For  $G = \mathrm{GL}(n)$ , the converse is also true. This is just the character theory. For other  $G$ , this is in general not the case.

Now explain the concept of “globally conjugate in image”. It arise from the following subtle situation. Given  $\rho$  and  $\rho'$  as before, if  $\rho(H)$  and  $\rho'(H)$  are conjugate in  $G$  (and hence also called globally conjugate), then we don't necessarily have  $\rho$  and  $\rho'$  are globally conjugate. In fact, we have the following easy lemma:

**Lemma 2.1.** *Let  $\rho$  and  $\rho': H \rightarrow G$  as before. If they are globally conjugate in image, then there is a  $\lambda \in \mathrm{Aut}(H)$  such that  $\rho$  and  $\rho' \circ \lambda$  are conjugate.*

$$\begin{array}{ccc} H & \xrightarrow{\rho} & G \\ \downarrow \lambda & & \downarrow c_g \\ H & \xrightarrow{\rho} & G \end{array}$$

*Proof:* By modifying by a conjugation, may assume that  $\rho(H)$  and  $\rho'(H)$  have the same image. Then  $\rho$  and  $\rho'$  must differ by a conjugation.  $\square$

To measure the distance among these concepts, we introduce, like in [29], the multiplicity numbers.

*Definition–Multiplicities:* We denote  $M(\rho; G)$  the set of equivalent classes of  $\rho': H \rightarrow G$  which is locally conjugate to  $\rho$ , modulo the global conjugacy, and  $M'(\rho; G)$  the set of equivalent classes of  $\rho': H \rightarrow G$  which is locally conjugate to  $\rho$ , modulo the global conjugacy in image. Moreover, denote  $m(\rho; G)$  and  $m'(\rho; G)$  as the cardinalities of  $M(\rho; G)$  and  $M'(\rho; G)$  respectively.

Here are examples and some of the known facts. Unless specified,  $H$  denotes a complex reductive group.

*Example:* When  $G = \mathrm{GL}(n, \mathbb{C})$  and  $H$  reductive and not necessarily connected,  $m(\rho; G) = m'(\rho; G) = 1$ .

*Example:* When  $G = \mathrm{SL}(N, \mathbb{C}), \mathrm{Sp}(2N, \mathbb{C}), \mathrm{O}(N, \mathbb{C}), \mathrm{SO}(2N+1, \mathbb{C})$ , and  $H$  reductive, not necessarily connected, then  $m(\rho; G) = m'(\rho; G) = 1$ . This is well known for a long time. To make this note complete, we include the whole purely algebraic proof in Section 2.2.

*Fact:* As  $M'(\rho; G)$  is a quotient set of  $M(\rho; G)$ ,  $m'(\rho; G) \leq m(\rho; G)$  with the equality holds if  $\mathrm{Aut}(H) = \mathrm{Inn}(H)$ , i.e.,  $H$  has no outer automorphism.

*Example:* When  $G = \mathrm{SO}(2N, \mathbb{C})$ , then  $m(\rho; G)$  and  $m'(\rho; G)$  are 1 or 2. One purpose of this note is to discuss this explicitly when  $H$  is connected. In particular, when  $G = \mathrm{SO}(10, \mathbb{C})$ ,  $H = \mathrm{SO}(5, \mathbb{C})$  and  $\rho = \Lambda^2$ , the exterior square map, then  $m'(\rho; G) = 2$ . This is the simplest case when  $H$  is connected and  $m'(\rho; G) = 2$ .

*Example:* When  $G = \mathrm{PGL}(N, \mathbb{C})$ ,  $m(\rho; G), m'(\rho; G)$  might not be 1 ([4], [15]). However, when  $H$  is connected,  $m(\rho; G) = m'(\rho; G) = 1$ . See Section 2.3. .

*Examples:* When  $G = F_4, E_6, E_7$  or  $E_8$  (any isogenous form),  $H$  is finite,  $m(\rho; G)$  and  $m'(\rho; G)$  might be also greater than 1 (see [16], [17]).

*Example:* When  $G = E_8(\mathbb{C})$  and  $H = \mathrm{PGL}(3, \mathbb{C})$ , there is an embedding  $\rho: H \hookrightarrow G$  such that  $m(\rho; G) = 2$  ([21], [22], and also Subsection 3.2)

*Fact:* for general  $G$  and reductive  $H$ , we have  $m(\rho; G)$  and  $m'(\rho; G)$  are finite ([7], [8], [29], [30], [31]).

## 2.2. Structure Theory: Self-dual Representations.

In this part, we study the case when  $G$  is classical and  $H$  is reductive, not necessarily connected. The main results of this part are Theorem 2.2, Corollary 2.4, Theorem 2.5. For review of the concepts, see the discussion before the proof of Theorem 2.2.

**Theorem 2.2.** *Let  $G = \mathrm{O}(N, \mathbb{C})$ ,  $\mathrm{SO}(2N+1, \mathbb{C})$  or  $\mathrm{Sp}(2N, \mathbb{C})$ ,  $H$  complex reductive, and  $\rho, \rho': H \rightarrow G$  two homomorphisms. Then if  $\rho, \rho'$  are locally conjugate, then they are globally conjugate. Hence,  $m(\rho; G) = m'(\rho; G) = 1$ . In particular, two equivalent orthogonal / symplectic representations of  $H$  are isometric.*

Before we start the proof, we list something here.

**Proposition 2.3.** *When  $G = \mathrm{GL}(N, \mathbb{C}), \mathrm{SL}(N, \mathbb{C})$ ,  $H$  reductive and  $\rho: H \rightarrow G$  a rational representation, then  $m(\rho; G) = m'(\rho; G) = 1$ .*

□

**Corollary 2.4.** *When  $G = \mathrm{SO}(2N, \mathbb{C})$ ,  $H$ ,  $\rho$  as before, then  $m(\rho; G)$  and  $m'(\rho; G)$  are 1 or 2. In particular, if  $\rho$  and  $\rho'$  are locally conjugate, then either they are globally conjugate, or they differ by an outer automorphism of  $\mathrm{SO}(2N, \mathbb{C})$ .*

*Proof by using Theorem 2.2:*

Let  $\tilde{G} = \mathrm{O}(2N, \mathbb{C})$ ,  $I: G \hookrightarrow \tilde{G}$  the natural embedding, and  $\rho': H \rightarrow G$  be any homomorphism which is locally conjugate to  $\rho$ . Then  $I \circ \rho$  and  $I \circ \rho'$  are also locally conjugate. By Theorem 2.2,  $I \circ \rho$  and  $I \circ \rho'$  are globally conjugate in  $\tilde{G}$ . Let  $g \in \mathrm{O}(2N, \mathbb{C})$  be such that  $I\rho'(x) = g(I \circ \rho(x))g^{-1}$  for all  $x \in G$ . Then we have  $\det(g) = \pm 1$ . Let  $c_g$  be the conjugation by  $g$ . Then  $c_g$  stabilizes  $I(G)$  and hence leads to an automorphism of  $G$ , also called  $c_g$ , and  $\rho' = c_g \circ \rho$ . If  $\det(g) = 1$ , then  $c_g$  is inner; If  $\det(g) = -1$ , then  $c_g$  is an outer automorphism. Hence we have the corollary.

□

Now we analyze the structure of the self-dual representation. First recall some notations and concepts.

Recall that given a vector space  $W$  together with a bilinear form  $B$  (not necessarily with a group action), a subspace  $W'$  is said to be *non-degenerate* (resp. *totally isotropic*) if when  $W'$ , together with the bilinear form  $B|_{W'}$ , is non-degenerate (resp. totally isotropic). (Recall that a space  $W$  with a bilinear form  $B$  is said to be *totally isotropic* if  $B(v, w) = 0$  for all  $v, w \in W$ .)

Recall that, given a finite-dimensional complex representation  $(\rho, V)$  of a group  $H$ , the *contragredient*  $(\rho^\vee, V^\vee = V^*)$  of  $\rho$  is defined as,  $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$  and  $G$  acts as the following:  $(\rho^\vee(g)(\lambda))(v) = \lambda(\rho(g^{-1})v)$  for  $\lambda \in V^*, v \in V, g \in H$ . A representation  $(\rho, V)$  of  $H$  is said to be *self-dual* if  $\rho^\vee$  is equivalent to  $\rho$ , which is equivalent to say that, there is a non-degenerate bilinear form  $B : V \times V \rightarrow \mathbb{C}$  which is  $H$ -equivariant, i.e.,  $B(\rho(g)v, \rho(g)w) = B(v, w)$  for  $v, w \in V$ . Moreover, it is well known that, if  $\rho$  is irreducible and self-dual, then such  $B$  must be unique up to scalar (Schur's Lemma, see Lemma 2.6), and moreover must be symmetric or alternating. We also say that in this case  $\rho$  is *of orthogonal type* (resp. *of symplectic type*) or just *orthogonal* (resp. *symplectic*) if  $B$  is symmetric (resp. alternating). A  $H$ -invariant subspace or  $H$ -subspace of  $V$  is a subspace which is stable under the action of  $H$ .

Note that when we talk about the *orthogonal/symplectic structure* of a self-dual representation  $(\rho, V)$  of  $H$  we also means together a non-degenerate  $H$ -invariant symmetric/alternating bilinear form  $B$ , which is not unique in general but is unique up to scalar when  $\rho$  is irreducible. In this case,  $\rho$  is induced by a homomorphism  $\rho_0: H \rightarrow G$  where  $G = \text{O}(N), \text{Sp}(2N)$  (or  $\text{SO}(N)$  if the image of  $\rho$  happens to lie inside  $\text{SL}(V)$ ) through the standard representation  $G \xrightarrow{\cong} \text{O}(V, B) \hookrightarrow \text{GL}(V)$  where  $\text{O}(V, B) := \{g \in \text{GL}(V) \mid B(\rho(g)v, \rho(g)w) = B(v, w) \forall v, w \in V\}$ .  $\rho_0$  or  $\rho$  is called *essential* if any totally isotropic  $H$ -invariant subspace of  $V$  must be trivial, and  $\rho_0$  or  $\rho$  is called *stable* if  $\rho$  is irreducible. Of course if  $\rho$  or  $\rho_0$  is stable then it is essential. Moreover, it is well known that  $\rho_0$  is essential if and only if the image of  $\rho_0$  is not contained in any proper parabolic subgroup of  $G$ . So the concept “essential” should be the analogue of “irreducible” to the case  $G$  classical.

*Proof of Theorem 2.2:*

Let  $V$  and  $V'$  be finite dimensional self-dual complex representations of  $H$ , and  $B$  and  $B'$  are orthogonal/symplectic structures on  $V$  and  $V'$  induced by  $\rho$  and  $\rho'$  respectively. As  $\rho$  and  $\rho'$  are locally conjugate,  $V$  and  $V'$  must be equivalent as  $H$ -representations (Proposition 2.3). Now our theorem follows from the following claim: There exists an isomorphism  $T : V \rightarrow V'$  which is  $H$ -equivariant and  $B = B'^T$  where  $B'^T(u, v) = B'(Tu, Tv) \forall u, v \in V$ , i.e,  $T$  is an  $H$ -equivariant isometry.

Now we prove the claim: Let  $T_0 : V \rightarrow V'$  be an arbitrary  $H$ -equivariant isomorphism. It is easy to see that  $B'^{T_0}$  is also a non-degenerate  $H$ -invariant bilinear form on  $V$ . Through  $B$  and  $B'^{T_0}$  we get two  $H$ -representation isomorphisms from  $V$  to  $V^*$ . Thus there is a  $L \in \text{GL}(V)$  which is  $H$ -equivariant such that the following diagram commutes.

$$\begin{array}{ccc} V & \xrightarrow{i_{B'^{T_0}}} & V^* \\ \downarrow L & & \downarrow = \\ V & \xrightarrow{i_B} & V^* \end{array}$$

where  $i_B: V \rightarrow V^*$  is the isomorphism induced by  $B$ , i.e.,  $i_B(w)(v) = B(v, w) \forall v, w \in V$ . In particular,  $B'^{T_0} = B^L$ . Hence  $B = B'^{T_0 \circ L^{-1}}$ , and  $T = T_0 \circ L^{-1}$  is what we want.

□



**Theorem 2.5.** *Let  $V$  be a finite dimensional self-dual complex representation of a complex reductive group  $H$  with  $B$  a non-degenerate  $H$ -invariant bilinear form on  $V$  which is either symmetric or alternating. Then we have the following.*

- (1)  *$V$  is a semisimple representation of  $H$ , i.e.,  $V$  is a direct sum of irreducible representations.*
- (2) *Each irreducible  $H$ -subspace of  $V$  is either non-degenerate or totally isotropic.*
- (3) *Let  $V_\sigma$  be the  $\sigma$ -isotypical component of  $V$ , i.e., the sum of all irreducible subspace equivalent to  $\sigma$ , and let  $W_\sigma = V_\sigma$  if  $\sigma$  is self-dual, and  $V_\sigma + V_{\sigma^\vee}$  if  $\sigma$  is not self-dual. Then  $V$  is the direct sum of  $W_\sigma$ , and the direct sum is orthogonal, and each  $W_\sigma$  is non-degenerate.*
- (4) *If  $\sigma$  is not self-dual, then  $W_\sigma = V_\sigma + V_{\sigma^\vee}$  gives rises to a complete polarization of  $W_\sigma$ . If  $\sigma$  is self-dual, then  $W_\sigma = V_\sigma \cong W_{\sigma,0} \otimes W'_\sigma$  where  $W_{\sigma,0} \cong \sigma$  and  $H$  acts on  $W'_\sigma$  trivially. Moreover, we can endow  $W'$  an orthogonal/symplectic structure such that the isomorphism is also an isometry.*

**Lemma 2.6. (Schur's Lemma)** *Let  $V, W$  be two complex finite-dimensional irreducible representations of a group  $H$ , and  $T \in \text{Hom}_H(V, W)$ . Then*

- (1)  *$T$  is either 0 or an isomorphism.*
- (2) *If  $V = W$ , then  $T = \text{cid}_V$ , a scalar multiplication.*
- (3) *If  $B, B'$  are two non-degenerate  $H$ -invariant bilinear forms on  $V$ , and  $B$  is non-degenerate, then  $B' = cB$  for some  $c \in \mathbb{C}$ .*

□

*Proof of Theorem 2.5:*

(1) follows from Proposition 2.3.

(2): for each irreducible  $H$ -subspace  $W$ ,  $W \cap W^\perp$  is also  $H$ -stable, where  $W^\perp = \{w' \in V \mid B(w, w') = 0 \ \forall w \in W\}$ . As  $W$  is irreducible,  $W \cap W^\perp = W$  or 0. Hence the assertion.

(3): First, let  $W$  and  $W'$  be two irreducible subspaces of  $V$ , and assume that  $W'$  is not congruent to  $W^\vee$ , then we claim that  $W$  and  $W'$  are orthogonal. In fact, as  $W'$  is not equivalent to  $W^* \cong W^\vee$ , the map  $W' \rightarrow W^*$ ,  $w \mapsto B(\cdot, w)$ , which is  $H$ -equivariant, must be 0 by Lemma 2.6. Hence the claim.

Thus,  $V_\sigma$  and  $V_{\sigma'}$  are orthogonal if  $\sigma'$  is not equivalent to  $\sigma^\vee$ , and  $W_\sigma$  and  $W_{\sigma'}$  are orthogonal if  $\sigma'$  is not equivalent to  $\sigma$  or  $\sigma^\vee$ .

Thus  $V$  is the orthogonal direct sum of  $W_\sigma$  when  $\sigma$  runs through all pairs  $\{\sigma, \sigma^\vee\}$  of irreducible representations of  $H$ . Moreover, all  $W_\sigma$  are non-degenerate since  $V$  itself is non-degenerate.

(4): For  $\sigma$  not self-dual,  $V_\sigma$  and  $V_{\sigma^\vee}$  form a complete polarization of  $W_\sigma$  since each irreducible subspace of type  $\sigma$  must be totally isotropic.

For  $\sigma$  self-dual, want to analyze  $W_\sigma = V_\sigma$ . As it is also non-degenerate from (3), may assume  $V = V_\sigma$ , with the non-degenerate  $H$ -invariant bilinear form  $B$ , either symmetric or alternating.

Let  $W = W_{\sigma,0}$  be an irreducible representation of  $H$  of type  $\sigma$ , with a non-degenerate  $H$ -invariant symmetric/alternating form  $B_0$ . Thus  $V \cong W \otimes W'$  as  $H$ -representations as  $V = V_\sigma$  is  $\sigma$ -isotypical, where the action of  $H$  on the right side is on the first factor. We identify both sides for such isomorphism.

We claim that we can endow a non-degenerate bilinear form  $B'$  on  $W'$ , such that

$$B(u \otimes u', v \otimes v') = B_0(u, v)B'(u', v')$$

for each  $u, v \in W$  and  $u', v' \in W'$ .

Granting this claim, then set  $W'_\sigma = W'$ , and moreover  $B'$  is symmetric or alternating according to the types of  $B$  on  $V \cong W \otimes W'$  and  $B_0$  on  $W$ . Hence the second assertion of (4) follows. The rest assertions follow also.

Now come to the claim. First, for each pair  $(u, v) \in W^2$  that  $B_0(u, v) \neq 0$ , define a bilinear form  $B'_{(u,v)}$  on  $W'$  such that

$$B(u \otimes u', v \otimes v') = B_0(u, v)B'_{(u,v)}(u', v')$$

for each  $u', v' \in W'$ . The subscript for  $B'_{(u,v)}$  is present since the definition depends on the choice of  $(u, v)$ , and we will see that finally the subscript can be dropped.

We want to prove: (a)  $B'_{(u,v)}$  is independent of the choice of  $(u, v)$ . (b)  $B' = B'_{(u,v)}$  is non-degenerate, and the claim holds for such  $B'$ .

Assume that  $B'_{(u_0, v_0)}(u', v') \neq 0$ ,  $u_0, v_0 \in W$ ,  $u', v' \in W'$ . Such  $u_0, v_0, u', v'$  exist as  $B$  is non-degenerate and there are  $u, v \in W$ ,  $u', v' \in W'$  such that  $B(u \otimes u', v \otimes v') \neq 0$ .

Fix  $u'$  and  $v'$ , and put  $\tilde{B}_0$  a bilinear form on  $W$  such that  $B(u \otimes u', v \otimes v') = \tilde{B}_0(u, v)B'_{(u_0, v_0)}(u', v')$ . Then  $\tilde{B}_0$  is also  $H$ -invariant. As  $W$  is irreducible, then  $\tilde{B}_0 = CB_0$  for some complex number  $C$  by Lemma 2.6. As  $\tilde{B}_0(u_0, v_0) = B_0(u_0, v_0)$ ,  $\tilde{B}_0 = B_0$ . Thus we have  $B(u \otimes u', v \otimes v') = B_0(u, v)B'_{(u_0, v_0)}(u', v')$  for ALL  $u, v \in W$ . So for all  $u, v \in W$  with  $B_0(u, v) \neq 0$ ,  $B'_{(u,v)}(u', v') = B'_{(u_0, v_0)}(u', v') \neq 0$ . Since  $B'_{(u,v)}(u', v')$ , as a quotient  $B(u \otimes u', v \otimes v')$  by  $B_0(u, v)$ , is a rational function of  $W^2 \times W'^2$ ,  $B'_{(u,v)}(u, v)$  must be independent of the choice of  $(u, v)$  and hence (a).

Let  $B' = B'_{(u,v)}$  for any choice of  $(u, v)$ . Then  $B'$  is a bilinear form on  $W'$ , and then (b) and the claim follow easily.

□

For later use, we quote the following.

**Proposition 2.7.** *Let  $(\rho, V)$  be a self-dual finite dimensional representation of a complex reductive group  $H$  with a  $H$ -invariant symmetric/alternating bilinear form  $B$ . Then  $\rho$  is essential if and only if  $\rho$  is a direct sum of inequivalent self-dual representations. Let  $V_\sigma$  be as in Theorem 2.5 for each irreducible representation  $\sigma$ . Then in this case, either  $V_\sigma = 0$  or  $V_\sigma$  is irreducible and non-degenerate.*

*Proof:* All notations are the same as in Theorem 2.5.

(1) Assume that  $\rho$  is essential. By Theorem 2.5 (2), all irreducible constituents  $\sigma$  of  $V$  must be self-dual, and by Theorem 2.5 (4), we have  $V = \bigoplus_\sigma V_\sigma$ , while

$V_\sigma \cong W_{\sigma,0} \otimes W'_\sigma$  being the isometry, where  $\sigma$  runs through all irreducible self-dual constituents  $\sigma$  of  $V$ . If for some  $\sigma$ ,  $W'_\sigma$  has dimension  $\geq 2$ , then  $W'$  must have an isotropic vector. Then  $W_{\sigma,0} \otimes \mathbb{C}w'$  is a totally isotropic  $H$ -subspace of  $V_\sigma = W_{\sigma,0} \otimes W'_\sigma$ . Through the isometry, this gives rise to a nontrivial totally isotropic  $H$ -subspace. This contradicts the assumption that  $\rho$  is essential. Hence, Each  $W'_\sigma$  must be 1-dimensional and  $V_\sigma$  must be irreducible. In particular,  $V$  is a direct sum of inequivalent irreducible non-degenerate  $H$ -subspaces (Theorem 2.5 (3)).

(2) Assume now that  $V$  is a direct sum of inequivalent irreducible non-degenerate  $H$ -subspaces, and  $W$  a totally isotropic  $H$ -subspace. Want to prove that  $W = 0$ . Otherwise, let  $W'$  is an irreducible  $H$ -subspace of  $W$ , and want to get the contradiction. By Theorem 2.5,  $W' \subset W_\sigma$  for some self-dual  $\sigma$ . By the assumption, for each constituent  $\sigma$ ,  $\sigma$  is self-dual and  $W_\sigma = V_\sigma$  is itself irreducible. Thus  $W' = V_\sigma$  is non-degenerate (Theorem 2.5), and hence  $W \supset W'$  can't be totally isotropic. Contradiction. Done.

□

### 2.3. About Isogenous Forms.

From now on we focus on the case when  $H$  a connected reductive group, still,  $\rho$  and  $\rho'$  are homomorphisms from  $H$  to  $G$ . To prove Theorem A, we need to study the relations between the multiplicities and the isogenous forms. The main results are Theorem 2.8 and Theorem 2.14, and at the end of this subsection we will prove Theorem A.

First some definitions. Unless specified, all groups involved in this section are connected complex reductive. We say that  $\pi : \tilde{G} \rightarrow G$  is a *finite central isogeny* if  $\pi$  is a finite homomorphism with its kernel central finite. In this case, we also say that  $\tilde{G}$  is a *finite central isogenous cover* (or just *isogenous cover*) of  $G$ . It is not hard to see that the composition of two finite central isogenies of complex reductive groups is still a finite central isogeny. Moreover, two connected groups  $G$  and  $G'$  are said to be *isogenous* or *of the same isogenous form* if they share a common finite central isogenous cover, i.e., there is a  $G''$  such that both  $G'' \rightarrow G$  and  $G'' \rightarrow G'$  are finite central isogenies. In this case we also say that  $G'$  is an *isogenous form* of  $G$ . It is known that two connected reductive groups are isogenous if and only if they share the same Lie algebra.

We say that  $\tilde{\rho} : \tilde{H} \rightarrow \tilde{G}$  is a *finite central isogenous cover* of  $\rho : H \rightarrow G$  if the following diagram commutes

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\tilde{\rho}} & \tilde{G} \\ \downarrow \pi_H & & \downarrow \pi_G \\ H & \xrightarrow{\rho} & G \end{array}$$

where  $\pi_H$  and  $\pi_G$  are also finite central isogenies. We say that  $\rho : H \rightarrow G$  and  $\rho_1 : H_1 \rightarrow G_1$  are *isogenous* if they possess the same finite central isogenous covering. In this case,  $H$  and  $H_1$ ,  $G$  and  $G_1$  are of the same isogenous form.

**Theorem 2.8.** *Let  $\rho : H \rightarrow G$  and  $\rho' : H' \rightarrow G'$  be two isogenous homomorphisms of connected complex reductive groups. Then  $m(\rho; G) = m(\rho'; G')$  and  $m'(\rho; G) = m'(\rho'; G')$ .*

This theorem enable us to reduce the case to a convenient isogenous form of  $G$  in our discussion. For the definition of multiplicities, see Section 2.1.

According to the definition, it suffices for us to prove the case when one of  $\rho'$  and  $\rho$  is a finite central isogenous cover of another.

**Proposition 2.9.** *Let  $H$  be a connected reductive group and  $\pi : \tilde{G} \rightarrow G$  a finite central isogeny of connected complex reductive algebraic groups. Given a homomorphism  $\rho$ , we can lift  $\rho$  to  $\tilde{\rho} : H \rightarrow \tilde{G}$  such that  $\rho = \pi \circ \tilde{\rho}$  if and only if we can lift  $\rho|_T$  where  $T$  is a maximal torus of  $H$ .*

$$\begin{array}{ccccc}
 & & & & \tilde{G} \\
 & & & \nearrow \tilde{\rho} & \downarrow \pi \\
 & & T & \xrightarrow{\tilde{\rho}_T} & \\
 & \nearrow \tilde{\rho} & & & \\
 T & \hookrightarrow & H & \xrightarrow{\rho} & G
 \end{array}$$

In this case, such  $\tilde{\rho}$  is unique.

*Remark:* In this lemma, we call such  $\rho$  *liftable*.

*Proof:*

The only if part of the first assertion is not a problem. Once the first assertion done, the uniqueness is also easy to see due to the connectedness of  $H$ . Now we prove the if part. We proceed by cases.

Case (1):  $H$  is semisimple and simply connected. In this case, such  $\tilde{\rho}$  definitely exists, even without the assumption about  $\rho|_T$ .

Case (2):  $H$  is semisimple. Let  $\tilde{H}$  be the simply connected complex Lie group isogenous to  $H$ , and  $\pi_H : \tilde{H} \rightarrow H$ .

$$\begin{array}{ccccc}
 \tilde{T} & \hookrightarrow & \tilde{H} & \xrightarrow{\tilde{\tilde{\rho}}} & \tilde{G} \\
 \pi_T \downarrow & & \pi_H \downarrow & \nearrow \tilde{\rho} & \downarrow \pi \\
 T & \hookrightarrow & H & \xrightarrow{\rho} & G
 \end{array}$$

Here  $\tilde{T}$  is the inverse image of  $T$  via  $\pi_H$  which is again a maximal torus of  $\tilde{H}$ . From case (1),  $\tilde{\tilde{\rho}}$  exists.

To prove that  $\tilde{\rho}$  exists it suffices to show that  $\tilde{\tilde{\rho}}(\text{Ker}(\pi_H)) = 1$ . This follows as since  $\rho|_T$  is liftable, and hence  $\tilde{\tilde{\rho}}(\text{Ker}(\pi_T)) = 1$ , and moreover, as  $\pi_H$  is a finite central isogeny, its kernel must be semisimple and contained in all maximal tori of  $\tilde{H}$ , and hence  $\text{Ker}(\pi_T) = \text{Ker}(\pi_H)$ .

Case (3):  $H$  itself is a torus. This is trivial as  $H = T$  and  $\rho = \rho|_T$ .

As  $H$  is connected, we deduce that  $\tilde{\rho}$  is unique, for at least the last three cases, and we will use this in the last case.

Case (4): General case.  $H$  connected reductive.

$$\begin{array}{ccccc}
 & & \tilde{G} & & \\
 & \nearrow \tilde{\rho}_{H_0} & \downarrow \pi & \nwarrow \tilde{\rho}_{T_0} & \\
 H_0 & \xrightarrow{\pi_{H_0}} & H & \xleftarrow{\pi_{T_0}} & T_0
 \end{array}$$

Let  $H_0 = (H, H)$  be the derived group of  $H$ ,  $T_0 \subset T$  the maximal central torus of  $H_0$ , and  $\rho_{H_0} = \rho|_{H_0}$  and  $\rho_{T_0} = \rho|_{T_0}$ , and let  $T_1 = H_0 \cap T$  be also a maximal torus of  $H_0$ . As  $\rho|_T$  is liftable, so is  $\rho|_{T_1}$  and  $\rho_{T_0}$ . Applying Case (2) (while  $H_0 = (H, H)$  is semisimple) and (3), we get, both  $\rho_{H_0}$  and  $\rho_{T_0}$  are liftable. Now define  $\tilde{\rho} : H \rightarrow G$  as

$$\tilde{\rho}(h_0 t_0) = \tilde{\rho}_{H_0}(h_0) \tilde{\rho}_{T_0}(t_0) \quad \forall h_0 \in H_0, t_0 \in T_0$$

We claim that it is well defined, and is a homomorphism.

If  $h_0 t_0 = 1$ , then  $h_0, t_0 \in Z(H_0) \subset T_1$ , and in particular, lies in  $T$ . Thus by the uniqueness of  $\tilde{\rho}_T$ ,  $\tilde{\rho}_{H_0}$  and  $\tilde{\rho}_{T_0}$  and  $\tilde{\rho}_{T_1}$  where  $\tilde{\rho}_{T_1} = \tilde{\rho}_T|_{T_1}$  is the lift of  $\rho_{T_1}$ , we have

$$\begin{aligned}
 \tilde{\rho}(h_0 t_0) &= \tilde{\rho}_{H_0}(h_0) \tilde{\rho}_{T_0}(t_0) && \text{(Definition.)} \\
 &= \tilde{\rho}_{T_1}(h_0) \tilde{\rho}_{T_1}(t_0) && (h_0 \in T_1 \text{ and } T_1 \subset H_0) \\
 &= \tilde{\rho}_{T_1}(h_0 t_0) = 1 && h_0 t_0 = 1
 \end{aligned}$$

As  $H_0$  and  $T_0$  commute,  $\tilde{\rho}$  is well defined. Now it is routine to check that  $\tilde{\rho}$  is a homomorphism and is a lift of  $\rho$ , and finally is unique.

□

**Lemma 2.10.** *Let  $T$  be a complex torus. Then  $T$  has a generic point, i.e., a point that generates a (Zariski or topologically) dense subset of  $T$ . the set of generic points of  $T$  is (Zariski or topologically) dense.*

□

**Lemma 2.11.** *Let  $\rho, \rho' : H \rightarrow G$  be two homomorphisms of complex connected reductive groups. Let  $T$  be maximal torus of  $H$ . Then  $\rho$  and  $\rho'$  are locally conjugate, if and only if  $\rho|_T, \rho'|_T : T \rightarrow G$  are globally conjugate.*

*Proof:*

(a) If part: Assume that  $\rho|_T$  and  $\rho'|_T$  are globally conjugate. From linear algebraic group theory,  $H^{ss}$ , the set of semisimple points of  $H$ , is Zariski dense in  $H$ , and contains an open subset of  $H$ , and moreover each semisimple element of  $H$  is conjugate to an element of  $T$ . Thus for each  $h \in H^{ss}$ ,  $\rho(h)$  and  $\rho'(h)$  are conjugate in  $G$ , and hence  $\rho$  and  $\rho'$  are locally conjugate.

(b) Only if part: Assume that  $\rho$  and  $\rho'$  are locally conjugate. By the definition,  $\rho(h)$  and  $\rho'(h)$  are conjugate in  $G$  for  $h$  in a (Zariski) dense open subset  $W$  in  $H$ . Since  $H^{ss}$  is Zariski dense in  $H$ ,  $W \cap H^{ss} \neq \emptyset$  and hence there is a maximal torus  $T_0$  which intersects  $W$ . As  $W$  is Zariski open dense in  $H$ ,  $W \cap T_0 \neq \emptyset$  must be also

Zariski open dense in  $T_0$ . From Lemma 2.10, there is a  $t_0 \in T_0 \cap W$  which is generic. Hence  $\rho|_{T_0}$  and  $\rho'|_{T_0}$  must be globally conjugate. The maximal tori  $T$  and  $T_0$  are conjugate (Well known!), and hence  $\rho|_T$  and  $\rho'|_T$  must be globally conjugate.

□

**Corollary 2.12.** *Let  $H, G, \tilde{H}, \tilde{G}$  be connected complex reductive groups,  $\pi_H : \tilde{H} \rightarrow H, \pi_G : \tilde{G} \rightarrow G$  finite central isogenies,  $\rho, \rho' : H \rightarrow G$  be homomorphisms. Assume*

(1)  $\rho$  and  $\rho'$  are locally conjugate.

(2)  $\rho$  lifts to  $\tilde{\rho}$  (see the diagram).

Then  $\rho'$  also lifts to  $\tilde{\rho}'$ , and the lift is unique.

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\rho, (? \tilde{\rho}')} & \tilde{G} \\ \downarrow \pi_H & & \downarrow \pi_G \\ H & \xrightarrow{\rho, \rho'} & G \end{array}$$

*Proof:* Let  $\tilde{T}$  be a maximal torus of  $\tilde{H}$  and  $T = \pi_H(\tilde{T})$ .

Consider  $\rho \circ \pi_H$ . As  $\rho$  lifts  $\tilde{\rho}$ ,  $\rho \circ \pi_H = \pi_G \circ \tilde{\rho}$ . Hence  $\rho|_T \triangleq (\rho \circ \pi_H)|_T$  is also liftable.

As  $\rho$  and  $\rho'$  are locally conjugate, so is  $\rho \circ \pi_H$  and  $\rho' \circ \pi_H$ . Pick In particular,  $\rho|_T$  and  $\rho'|_T \triangleq \rho' \circ \pi_H$  are globally conjugate (Lemma 2.11). Thus,  $\rho'|_T$  is also liftable.

$$\begin{array}{ccccc} \tilde{T} & \hookrightarrow & \tilde{H} & \xrightarrow{\tilde{\rho}'} & \tilde{G} \\ & \searrow & \downarrow \pi_H & & \downarrow \pi_G \\ & & H & \xrightarrow{\rho'} & G \end{array}$$

Then by Proposition 2.9,  $\rho' \circ \pi_H$  is liftable. This gives rise to the existence of  $\tilde{\rho}'$ . The uniqueness now follows easily from Proposition 2.9.

□

**Theorem 2.13.** *Let  $H, G, \tilde{H}, \tilde{G}$  be connected complex reductive groups,  $\pi_H : \tilde{H} \rightarrow H, \pi_G : \tilde{G} \rightarrow G$  finite central isogenies,  $\rho, \rho' : H \rightarrow G$  be homomorphisms which lift to  $\tilde{\rho}, \tilde{\rho}'$  respectively.*

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\rho, (? \tilde{\rho}')} & \tilde{G} \\ \downarrow \pi_H & & \downarrow \pi_G \\ H & \xrightarrow{\rho, \rho'} & G \end{array}$$

Then  $\tilde{\rho}$  and  $\tilde{\rho}'$  are locally conjugate / globally conjugate / globally conjugate in image if and only if so are  $\rho$  and  $\rho'$ .

*Proof:* We proceed in steps.

Step 1:  $\rho, \rho'$  are globally conjugate  $\implies \tilde{\rho}, \tilde{\rho}'$  are globally conjugate.

Let  $g \in G$  be such that  $\rho' = c_g \circ \rho$  where  $c_g : x \mapsto gxg^{-1}$  is the conjugation. Let  $\tilde{g} \in \tilde{G}$  be a lift of  $g$ . Then both  $\tilde{\rho}'$  and  $c_{\tilde{g}} \circ \tilde{\rho}$  are lifts of  $\rho'$ . They must be equal (Proposition 2.9).

Step 2:  $\tilde{\rho}, \tilde{\rho}'$  are globally conjugate  $\implies \rho, \rho'$  are globally conjugate.

Easy.

Step 3:  $\rho, \rho'$  are globally conjugate in image  $\implies \tilde{\rho}, \tilde{\rho}'$  are globally conjugate in image.

Let  $g \in G$  be such that  $\rho'(H) = g\rho(H)g^{-1}$  and  $\tilde{g} \in \tilde{G}$  a lift of  $g$ . Then  $\tilde{\rho}'(H)$  and  $g\tilde{\rho}(H)g^{-1}$  are both connected subgroups of  $\pi_G^{-1}(\rho'(H))$ . As  $\pi_G$  is a finite map,  $\tilde{\rho}'(H)$ ,  $g\tilde{\rho}(H)g^{-1}$  and  $(\pi_G^{-1}(\rho'(H)))^\circ$  (the identity component of  $(\pi_G^{-1}(\rho'(H)))$ ), must possess the same dimension as  $\rho(H)$  and  $\rho'(H)$ . Thus these three subgroups must be equal.

Step 4:  $\tilde{\rho}, \tilde{\rho}'$  are globally conjugate in image  $\implies \rho, \rho'$  are globally conjugate in image.

Easy also.

Step 5:  $\rho, \rho'$  are locally conjugate  $\iff \tilde{\rho}, \tilde{\rho}'$  are locally conjugate.

Let  $T$  be a maximal torus of  $H$ .

$$\begin{aligned} \rho, \rho' \text{ are locally conjugate} &\iff \rho|_T, \rho'|_T \text{ are globally conjugate} && \text{(Lemma 2.11)} \\ &\iff \tilde{\rho}|_T, \tilde{\rho}'|_T \text{ are globally conjugate} && \text{(Step 1 \& 2)} \\ &\iff \tilde{\rho}, \tilde{\rho}' \text{ are locally conjugate} && \text{(Lemma 2.11)} \end{aligned}$$

□

Recall that  $M(\rho; G)$  is the set of equivalent classes of  $\rho' : H \rightarrow G$  which is locally conjugate to  $\rho$ , modulo the global conjugacy, and  $M'(\rho; G)$  is the set of equivalent classes of  $\rho' : H \rightarrow G$  which is locally conjugate to  $\rho$ , modulo the global conjugacy in image. Moreover,  $m(\rho; G)$  and  $m'(\rho; G)$  as the cardinalities of  $M(\rho; G)$  and  $M'(\rho; G)$  respectively.

**Theorem 2.14.** *Let  $H, G, \tilde{H}, \tilde{G}$  be connected complex reductive groups,  $\pi_H : \tilde{H} \rightarrow H$ ,  $\pi_G : \tilde{G} \rightarrow G$  finite central isogenies,  $\tilde{\rho}$  be a finite central isogenous cover of  $\rho$ .*

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\rho, (\tilde{\rho}')} & \tilde{G} \\ \downarrow \pi_H & & \downarrow \pi_G \\ H & \xrightarrow{\rho, (\rho')} & G \end{array}$$

*Then the lift of  $\rho' : H \rightarrow G$  to  $\tilde{\rho}' : \tilde{H} \rightarrow \tilde{G}$  define two well defined bijections from  $M(\rho, G)$  to  $M(\tilde{\rho}, \tilde{G})$  and from  $M'(\rho; G)$  to  $M'(\tilde{\rho}; \tilde{G})$  respectively. In particular,  $m(\rho; G) = m(\tilde{\rho}; \tilde{G})$  and  $m'(\rho; G) = m'(\tilde{\rho}; \tilde{G})$ .*

*Proof of Theorem 2.14 and Theorem 2.8*

Denote  $P(\rho; G)$  the set of homomorphism  $\rho' : H \rightarrow G$  that is locally conjugate to  $\rho$  and  $P(\tilde{\rho}; \tilde{G})$  similarly. As  $H$  is connected, by Corollary 2.12, we may lift  $\rho'$  to a unique  $\tilde{\rho}'$  (see the diagram) which gives rise to a well defined map

$$\Phi_\rho : P(\rho; G) \rightarrow P(\tilde{\rho}; \tilde{G})$$

*Claim:*  $\Phi_\rho$  is surjective. (In fact, it is bijective.)

Granting this claim, and by Theorem 2.13, such  $I_\rho$  induce two well defined bijections from  $M(\rho, G)$  to  $M(\tilde{\rho}, \tilde{G})$  and from  $M'(\rho; G)$  to  $M'(\tilde{\rho}; \tilde{G})$  with respectively. Hence we get the theorem.

Now we prove the claim. Assume that  $\tilde{\rho}'$  is given and it is locally conjugate to  $\tilde{\rho}$ . As  $\tilde{\rho}$  and  $\tilde{\rho}'$  are locally conjugate, so are  $\pi_G \circ \tilde{\rho}$  and  $\pi_G \circ \tilde{\rho}'$ , and hence they should share the same kernel. In particular, both should factor through the same quotient. Hence the claim and the theorem.

Now Theorem 2.8 follows directly with the discussion at the beginning of this subsection.

□

**Lemma 2.15.** *Let  $\pi_G : \tilde{G} \rightarrow G$  be a finite central isogeny of connected complex reductive groups. Given  $\rho : H \rightarrow G$  or  $\tilde{\rho} : \tilde{H} \rightarrow \tilde{G}$  which is a homomorphism, where  $H$  or  $\tilde{H}$  is complex connected reductive. we can complete the following diagram to make  $\tilde{\rho}$  a finite central isogenous cover of  $\rho$ .*

$$\begin{array}{ccc} \tilde{H} & \xrightarrow{\tilde{\rho}} & \tilde{G} \\ \downarrow \pi_H & & \downarrow \pi_G \\ H & \xrightarrow{\rho} & G \end{array}$$

Moreover, if  $G$  and  $G'$  be two complex reductive group and  $\rho : H \rightarrow G$  a homomorphism from a complex connected reductive group  $H$  to  $G$ , then there is a homomorphism  $\rho' : H' \rightarrow G'$  which is isogenous to  $\rho$ .

*Proof:* The second assertion follows from the first. Now we work on the first assertion.

First, if  $\tilde{\rho}$  is given, we can just take  $H = \tilde{H}$ ,  $\pi_H = \text{id}$  and  $\rho = \pi_G \circ \tilde{\rho}$ .

Next, if  $\rho$  is given, we want to get  $\tilde{\rho}$ . Let  $\tilde{H} = (H \times_G \tilde{G})^\circ$  where  $H \times_G \tilde{G} = \{(h, \tilde{g}) \in G \times \tilde{G} \mid \rho(h) = \pi_G(\tilde{g})\}$  and  $\tilde{H}$  its identity component, and  $\pi_H$  and  $\tilde{\rho}$  be the coordinate projection maps which are homomorphisms. Then we get the commutative diagram.

Now we claim:  $\pi_H$  is a finite central isogeny. In fact,  $\pi_H$  is surjective, since  $\pi_G$  is surjective, and for each  $h \in H$ , we can find  $\tilde{g}$  with  $\pi_G(\tilde{g}) = \rho(h)$ , and thus  $(h, \tilde{g}) \in \pi_H^{-1}(h)$ . Moreover,  $\pi_H$  is a finite central isogeny, since  $\text{Ker}(\pi_H) = 1 \times \text{Ker}(\pi_G)$  is finite and central in  $H \times_G \tilde{G}$ .

Done.

□



*Proof of Theorem A:*

Now let  $G' = \mathrm{GL}(N, \mathbb{C}), \mathrm{SO}(2N+1, \mathbb{C}), \mathrm{Sp}(2N, \mathbb{C}), \mathrm{SO}(2N, \mathbb{C})$ , and  $G$  an isogenous form of  $G'$ . Let  $\rho : H \rightarrow G$  be a homomorphism of complex connected reductive groups. Then by Lemma 2.15, there is a  $\rho' : H' \rightarrow G'$ , a homomorphism of complex connected reductive groups, which is isogenous to  $\rho$ . By Theorem 2.8,  $m(\rho, G) = m(\rho', G')$  and  $m'(\rho, G) = m'(\rho', G')$ . Theorem A follows now from Theorem 2.2 and Corollary 2.4.

□

### 3. MAIN THEOREM AND PROOFS

In this section, we mainly study the case  $G = \mathrm{SO}(2N, \mathbb{C})$ . In fact, we will get certain classification of LFMO-special representations. The definition occurs in both the introduction, and Subsection 3.1. In fact, we will prove Theorem B in Subsection 3.1 and Theorem C, Theorem D and Corollary E in Subsection 3.2.

#### 3.1. $\mathrm{SO}(2N)$ , Global VS Local, I.

Now again let  $H$  be a complex reductive group,  $G = \mathrm{SO}(2N, \mathbb{C})$ , and  $\rho : H \rightarrow G$  a homomorphism.

Recall some definitions, an element  $g \in \mathrm{SO}(2N, \mathbb{C})$  is said to *have eigenvalue 1 or  $-1$*  if view  $g$  as an element of  $\mathrm{GL}(2N, \mathbb{C})$  via the  $2N$ -dimensional standard representation.  $\rho$  (or the representation induced by  $\rho$ ) is said to *have weight 1* if when restricted to one of its maximal torus, the representation space contains a trivial subrepresentation as a constituent.

**Lemma 3.1.** *Let  $g \in \mathrm{SO}(2N, \mathbb{C})$  be a semisimple element and  $g' = \tau(g)$ . Then  $g$  and  $g'$  are conjugate if and only if  $g$  has eigenvalue  $\pm 1$ .*

*Proof:*

Let  $V$  be the  $2N$ -dimensional complex space with an orthogonal structure  $B$  and identify  $G = \mathrm{SO}(2N, \mathbb{C})$  with  $\mathrm{SO}(V, B)$ .

First we analyze the structure of  $V$  under the action of  $g$ . As  $g$  is semisimple,  $V$  is a direct sum of eigenspaces  $V_\lambda$  of  $g$ . Moreover,  $V_\lambda$  and  $V_{\lambda'}$  are orthogonal unless  $\lambda = \lambda'^{-1}$ , and moreover,  $V_{\lambda^{-1}} \cong V_\lambda^*$  as vector spaces through  $B$ .

Next the only if part. Assume that  $g$  and  $g'$  are conjugate. Then  $g' = \tau(g) = ygy^{-1}$  for some  $y \in \mathrm{SO}(V, B)$ . Thus the automorphism  $c_{y^{-1}} \circ \tau$  of  $\mathrm{SO}(V, B)$  fixes  $g$ . In particular, there is a  $x \in \mathrm{O}(V, B) - \mathrm{SO}(V, B)$  commutes with  $g$ . Thus  $x$  stabilizes all  $V_\lambda$ . As  $x \in \mathrm{O}(V, B)$ , the action of  $x$  on  $V_\lambda$  and  $V_{\lambda^{-1}}$  are contragredient to each other, and hence  $\det(x|_{V_{\lambda^{-1}}}) = (\det(x|_{V_\lambda}))^{-1}$ . So  $-1 = \det(x) = \det(x|_{V_1})\det(x|_{V_{-1}})$ . So  $V_1$  or  $V_{-1}$  must be nontrivial.

Conversely, assume  $g$  has an eigenvalue  $\epsilon = \pm 1$ . Under some basis,

$$g = \mathrm{Diag}(\alpha_1, \dots, \epsilon, \epsilon, \dots, \alpha_1^{-1})$$

and

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Then  $g$  commutes with the following  $z \in \mathrm{O}(V, B) - \mathrm{SO}(V, B)$ .

$$z = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Now  $g' = \tau(g)$  is conjugate to  $zgz^{-1} = g$ .

□

**Theorem 3.2.** *Let  $H$  be a complex reductive group and  $\rho : H \rightarrow G = \mathrm{SO}(2N, \mathbb{C}) \subset \Omega = \mathrm{O}(2N, \mathbb{C})$  be a homomorphism.  $\rho' = \tau \circ \rho$  for some outer automorphism  $\tau$  of  $G$ . Let  $T$  be a maximal torus of  $G$ . Then*

(A) *The following are equivalent if  $H$  is connected.*

(a-1)  $\rho$  and  $\rho'$  are locally conjugate.

(a-2)  $\rho|_T$  and  $\rho'|_T$  are locally conjugate.

(a-3)  $C_\Omega(\rho(T)) \not\subseteq G$ .

(a-4)  $\rho$  has weight 1.

(B) *The following are equivalent.*

(b-1)  $\rho$  and  $\rho'$  are globally conjugate.

(b-2)  $C_\Omega(\rho(H)) \not\subseteq G$ .

(b-3)  $\rho$  has an irreducible subrepresentation which is both odd dimensional and of orthogonal type.

(C) *The following are equivalent.*

(c-1)  $\rho$  and  $\rho'$  are globally conjugate in image.

(c-2)  $N_\Omega(\rho(H)) \not\subseteq G$ .

Recall that  $C_G(G_0)$  and  $N_G(G_0)$  denote the centralizer and the normalizer of  $G_0$  in  $G$ .

*Remark:* The equivalence of (a-1), (a-2) and (a-3) apply also for arbitrary connected group  $G$ . The equivalence of (b-1) and (b-2), (c-1) and (c-2) apply also for any group  $H$  and  $G = \mathrm{SO}(2N, \mathbb{C})$ .

*Proof:* Recall that we only assume that  $H$  is connected in (A). Moreover, let  $V$  be the representation space of  $\rho$  and  $B$  the non-degenerate  $H$ -invariant bilinear form on  $V$  induced by  $\rho$ . Let  $\Omega = O(V, B)$ .

(A) The equivalence of (a-1), (a-2) follow from Lemma 2.11. Recall that  $\rho'|_T = \tau \circ \rho|_T = c_x \circ \rho|_T$  for some  $x \in \Omega - G$ . Thus (a-3) is equivalent to the fact that  $\rho|_T = c_z \circ \rho|_T$  for some  $z \in \Omega - G$  which is globally conjugate to  $\rho'|_T = c_x \circ \rho|_T$  as  $xz^{-1} \in G$ . Hence this gives rise to (a-1)  $\leftrightarrow$  (a-3).

Let  $t$  be a generic point of  $T$ . Assume (a-3), we have  $\rho(t)$  commutes with  $z \in \Omega - G$ , Hence  $\rho(t)$  has eigenvalue  $\pm 1$  (Lemma 3.1). So  $V^T = V^{\langle t^2 \rangle} \neq 0$  where  $\langle t^2 \rangle$  is the group generated by  $t^2$  since  $t^2$  is also a generic point of  $T$ . This gives (a-4). So (a-3) implies (a-4).

Now assume (a-4), and let again  $t$  be a generic point of  $T$ . (a-4) implies that  $\rho(t)$  has eigenvalue 1. Thus  $\rho(t)$  commutes with some  $z \in \Omega - G$ . As  $t$  is a generic point of  $T$ ,  $z$  also commutes with  $\rho(T)$ , and this gives (a-3). So (a-4) implies (a-3).

(B) Recall that  $\rho' = \tau \circ \rho = c_x \circ \rho$  for some  $x \in \Omega - G$ . Thus (a-3) is equivalent to the fact that  $\rho = c_z \circ \rho$  for some  $z \in \Omega - G$  which is globally conjugate to  $\rho' = c_x \circ \rho$  as  $xz^{-1} \in G$ . Hence this gives rise to (b-1)  $\leftrightarrow$  (b-2).

Now apply our structure theory (Theorem 2.5). For each irreducible constituent  $\sigma$ , we have the isotypical component  $V_\sigma$  of  $V$  and  $W_\sigma = V_\sigma$  if  $\sigma$  is self-dual and  $V_\sigma \oplus V_{\sigma^\vee}$  otherwise. Then  $V$  is an orthogonal sum of  $W_\sigma$ . If  $\sigma$  is not self-dual, then  $W_\sigma = V_\sigma \oplus V_{\sigma^\vee}$  gives rise to a complete polarization. If  $\sigma$  is self-dual,  $V_\sigma \cong W_{\sigma,0} \otimes W'_\sigma$  where  $W_{\sigma,0}$  is equivalent to  $\sigma$ , and  $H$  acts on the first factor. Moreover, we have non-degenerate bilinear forms  $B_{\sigma,0}$ ,  $B'_\sigma$  on  $W_{\sigma,0}$  and  $W'_\sigma$  respectively such that the isomorphism is an isometry. Moreover  $B_{\sigma,0}$  and  $B'_\sigma$  are both symmetric or both alternating.

Assume (b-3), and let  $\sigma$  be a irreducible constituent of  $V$  which is odd dimensional of orthogonal type. Then  $W_\sigma = V_\sigma \neq 0$ ,  $W'_\sigma \neq 0$  is an orthogonal space. Choose an anisotropic vector  $w' \in W'_\sigma$  (namely,  $B'_\sigma(w', w') \neq 0$ ), and form  $W''$  in  $V$  from  $W_{\sigma,0} \otimes w'$  through the isometry. Then  $W''$  is a non-degenerate  $H$ -subspace of  $V$  equivalent to  $\sigma$ . Let  $A \in GL(V)$  be such that  $A|_{W''} = -\text{id}$  and  $A|_{W''^\perp} = 1$ . Then  $A$  is  $H$ -equivariant and  $A \in \Omega = O(V, B)$  so that  $A \in C_\Omega(\rho(H))$ . Moreover  $\det(A) = (-1)^{\dim(W'')} = (-1)^{\dim \sigma} = -1$  as  $\sigma$  is odd dimensional. Then  $A \notin G$ , and this gives (b-2). Hence (b-3) implies (b-2).

Now assume that (b-3) fails, namely all irreducible constituents of  $\rho$  are either not self-dual or even dimensional. We want to prove  $C_\Omega(\rho(H)) \subset G$  so that (b-2) fails.

Choose arbitrary  $T \in C_\Omega(\rho(H))$ . Then  $T$  is  $H$ -equivariant, and hence  $T$  stabilizes all  $V_\sigma$  and  $W_\sigma$ . Denote  $T_\sigma = T|_{W_\sigma}$ . We *claim* that  $\det(T_\sigma) = 1$ . Once we have the claim,  $T \in \Omega \cap \text{SL}(V) = G$ . Hence  $C_\Omega(\rho(H)) \subset G$ .

Assume first that  $\sigma$  is not self-dual. Then  $T_\sigma$  stabilizes  $V_\sigma$  and  $V_{\sigma^\vee}$ , which give rise to a complete polarization of  $W_\sigma$ . Through  $B$ ,  $V_{\sigma^\vee} \cong V_\sigma^*$ . As  $T$  is orthogonal, then under a choice of dual bases of  $V_\sigma$  and  $V_{\sigma^\vee}$ , the matrix representations of  $T$  on these two spaces are transpose inverse to each other. Hence  $\det(T_\sigma) = 1$ .

Assume now that  $\sigma$  is self-dual. Then  $\sigma$  must be even dimensional and  $V_\sigma = W_\sigma \cong W_{\sigma,0} \otimes W'_\sigma$ . As  $T$  is  $H$ -equivariant, then by Schur's lemma (Lemma 2.6 also, plus a good exercise in linear algebra), the action of  $T$  on  $V_\sigma$  is induced by  $1 \otimes T'_\sigma$  where  $T'_\sigma \in O(W'_\sigma, B'_\sigma)$ . In particular  $\dim(T'_\sigma) = \pm 1$  and

$$\dim(T_\sigma) = (\dim(T'_\sigma))^{\dim(W_{\sigma,0})} = (\pm 1)^{\dim(\sigma)} = 1$$

In all cases,  $\det(T_\sigma) = 1$ . So  $\det(T) = 1$  and  $T \in \Omega \cap \mathrm{SL}(V) = G$ . Thus (b-2) fails. So (b-2) implies (b-3).

(C)  $\rho$  and  $\rho' = \tau \circ \rho$  are globally conjugate in image in  $G$  if and only if  $\rho(H)$  and  $\tau(\rho(H))$  are conjugate, if and only if  $\rho(H)$  is stabilized by  $c_y \circ \tau$  for some  $y \in G$ . As  $\tau = c_x$  for some  $x \in \Omega - G$ , the last situation occurs if and only if  $H$  is normalized by some element in  $\Omega - G$ . So (c-1) and (c-2) are equivalent.

□

*Proof of Theorem B:*

Now (a) is Theorem 3.2 (A) (a-1)  $\leftrightarrow$  (a-4), (b) is Theorem 3.2 (B) (b-1)  $\leftrightarrow$  (b-3), (c) is Theorem 3.2 (A) (c-1)  $\leftrightarrow$  (c-2).

□

**Corollary 3.3.** *Let  $H$  be a complex reductive group and  $\rho : H \rightarrow G = \mathrm{SO}(2N, \mathbb{C})$  be a homomorphism. Then  $m(\rho; G), m'(\rho; G) = 1, 2$ . Moreover,  $m(\rho; G) = 2$  if and only if (A) holds and (B) fails, and  $m'(\rho; G) = 2$  if and only if (A) holds and (C) fails. Here (A), (B) and (C) are the any one of equivalent conditions of (A), (B) and (C) in Theorem 3.2. In particular, if  $\rho$  is stable, then  $m(\rho; G) = 2$  if and only if  $\rho$  has weight 1.*

*Proof:* The first condition is exactly Corollary 2.4, and the rest can be routinely checked easily.

□

### 3.2. $\mathrm{SO}(2N)$ , Local VS Global, II.

In this part, we focus on LFMO-special representation. Let  $H$  be a complex (reductive) group. Recall that a homomorphism  $\rho : H \rightarrow \mathrm{O}(N, \mathbb{C}), \mathrm{Sp}(2N, \mathbb{C})$  (or the induced self-dual representation) is said to be stable if the representation it induced is irreducible, and is said to be essential if it has no nonzero totally isotropic constituent. For definition, see the introduction, or Subsection 2.2 (after Corollary 2.3 and before the proof of Theorem 2.2).

Recall the definition. A homomorphism  $\rho : H \rightarrow \mathrm{SO}(2N, \mathbb{C})$  is said to be *LFMO-special* if, for  $\rho' = \tau \circ \rho$  for an outer automorphism  $\tau$  of  $\mathrm{SO}(2N)$ , the following conditions for  $\rho$  and  $\rho'$  hold.

(1)  $\rho$  is *essential*, namely, the image of  $\rho$  is not contained in any parabolic subgroup of  $\mathrm{SO}(2N)$ .

(2)  $\rho$  and  $\rho'$  are locally conjugate.

(3)  $\rho$  and  $\rho'$  are not globally conjugate in image.

In this case, we also say that the induced  $2N$ -dimensional representation is LFMO-special.

The name “LFMO” comes from the following: If  $\rho$  is LFMO-special, then  $\rho$  and  $\rho'$  will lead to failure of multiplicity one. That is also why we call this name (see [29]). In one word,  $\rho$  is LFMO-special if and only if  $\rho$  is essential and  $m'(\rho; \mathrm{SO}(2N, \mathbb{C})) = 2$ .

*Proof of Theorem C:* Easy right now.

According to our definition, the condition (1), (2) and (3) of Theorem C match the condition (1), (2) and (3) of our definition (where (1) from Proposition 2.7, (2) and (3) from Theorem 3.2, also see Corollary 3.3). Moreover, the last statement follows from Theorem 2.8 as when view  $\rho, \rho'$  as homomorphisms from  $H$  to  $G$ , they are isogenous since they are quasi-equivalent when viewed as complex representations.  $\square$

Now we focus on stable orthogonal representations and try to classify stable LFMO-special representations. Note that from Corollary 3.3, we have  $m(\rho; G) = 2$  if and only if  $\rho$  has weight 1, and now  $m'(\rho; G) = 2$  if and only if  $\rho$  is LFMO-special. As each  $H$  has a finite central isogenous cover  $T \times H_1 \times H_2 \times \dots \times H_r$  which is a decomposable group for  $T$  a torus, and  $H_i$  simple Lie group for each  $i$ ,  $\rho$  is isogenous to  $\rho'$  a homomorphism from  $H'$  to  $G$ . Theorem C, Theorem 2.8 and Corollary 3.3 enable us to reduce the classification problem to the case when  $H$  itself is decomposable. Moreover,  $\rho$  is also factorizable as a tensor product of self-dual representations, and the factorization is unique. and  $\rho|_T$  is trivial as  $T$  is central and  $\rho$  is stable (Lemma 2.6). We will observe this scenario (see Assumption A below), and replace each  $H_i$  by its isogenous form when convenient till the end of this section.

*Remark:* When  $H$  is decomposable,  $H$  is semisimple if and only if  $T = 1$ ; In addition,  $H$  is simply connected / adjoint if and only if so are  $H_i$  for all  $i$ .

**Lemma 3.4.** *With  $H$  decomposable as above,  $\rho$  has weight 1 if and only if so does  $\rho_i$  for each  $i \geq 1$ .*

$\square$

The lemma above plus Theorem C, enable us to reduce the problem to the study of the lift from automorphisms from  $\rho(H)$  to the automorphism of  $G$ .

**Lemma 3.5.** *Let  $H$  be a complex semisimple, simply connected or adjoint group, and  $\rho: H \rightarrow G = \mathrm{Sp}(2N, \mathbb{C})$  or  $\mathrm{SO}(N, \mathbb{C})$  then automorphisms of  $\rho(H)$  lift to those of  $H$ , i.e., given  $\phi \in \mathrm{Aut}(\rho(H))$ , there exists a  $\phi_0 \in \mathrm{Aut}(H)$ , such that  $\phi \circ \rho = \rho \circ \phi_0$ .*

$$\begin{array}{ccc} H & \xrightarrow{\rho} & \rho(H) \\ \downarrow \phi_0 & & \downarrow \phi \\ H & \xrightarrow{\rho} & \rho(H) \end{array}$$

This lemma enable us to reduce our problem to the study of  $\text{Aut}(H)$  in place of  $\text{Aut}(\rho(H))$ .

*Proof:* First case, assume that  $\text{Ker}(\rho)$  is finite.  $\rho$  is also a finite central isogeny from  $H$  to  $\rho(H)$ . Thus we get  $\phi_0 : H \rightarrow H$  as a lift of  $\phi \circ \rho : H \rightarrow \rho(H)$ , since when  $H$  is simply connected we can do this definitely (see the proof of Proposition 2.9), and when  $H$  is adjoint,  $\rho$  is an isomorphism.

In general, let  $N_0 = (\text{Ker}(\rho))^\circ$ . Then  $N_0$  is normal in  $H$ . As  $H$  is semisimple and simply connected / adjoint,  $H = N_0 \times H'$  for some  $H'$  which is also normal and semisimple. In fact both  $N_0$  and  $H'$  are product of simply connected / adjoint simple groups. Now  $\rho$  factors through  $H' \cong H/N_0$  and induces a finite central isogeny from  $H'$  to  $\rho(H)$ . Thus  $\phi \in \text{Aut}(\rho(H))$  lifts to  $\phi'_0 \in \text{Aut}(H')$  from the first case. The lemmas follows now easily when take  $\phi_0 = \text{id}_{N_0} \times \phi'_0$ .

□

Before we state our main results, we list some definition (which is also used in [28], [29]).

Let  $\rho_0 : H \rightarrow \text{GL}(V)$  be a finite dimensional representation, and  $V$  its representation space. We say that  $\phi \in \text{Aut}(H)$  is  $\rho_0$ -*liftable* if  $\rho_0$  and  $\rho_0 \circ \phi$  are equivalent, i.e., there is a  $T \in \text{GL}(V)$  such that  $\rho_0 \circ \phi = c_T \circ \rho_0$ , namely,

$$\begin{array}{ccc} H & \xrightarrow{\rho_0} & \text{GL}(V) \\ \downarrow \phi & & \downarrow c_T \\ H & \xrightarrow{\rho_0} & \text{GL}(V) \end{array}$$

and in this case we also say that  $\phi$  *lifts to  $T$  (through  $\rho$ )*.

Let  $\rho : H \rightarrow G = \text{SO}(N, \mathbb{C})$  or  $\text{Sp}(2N, \mathbb{C}) < \text{GL}(V)$  be a homomorphism with  $V$  its representation space, and also denote  $\rho$  as the representation. We say that  $\phi$  is  $\rho$ -*even* if  $G = \text{SO}(2N, \mathbb{C})$  or  $\text{Sp}(2N, \mathbb{C})$  and  $\phi$  lifts to some  $T \in G$ ,  $\phi$  is  $\rho$ -*odd* if  $G = \text{SO}(2N, \mathbb{C})$ , and  $\phi$  lifts to some  $T \in \Omega - G$  where  $\Omega = \text{O}(2N, \mathbb{C}) < \text{GL}(V)$ , and  $\phi$  is  $\rho$ -*neutral* if  $G = \text{SO}(2N+1, \mathbb{C})$  and  $\phi$  is  $\rho$ -liftable.

Also, recall that two representations  $\rho$  and  $\rho'$  of  $H$  and  $H'$  are *quasi-equivalent* if and only if there are finite central isogenies  $\iota : H'' \rightarrow H$  and  $\iota' : H'' \rightarrow H'$  such that as representations of  $H''$ ,  $\rho \circ \iota$  and  $\rho' \circ \iota'$  are conjugate.

**Proposition 3.6.** *Let  $H$  be a complex connected reductive group,  $\rho : H \rightarrow G = \text{Sp}(2N, \mathbb{C})$  or  $\text{SO}(N', \mathbb{C})$  be a stable homomorphism and  $\phi \in \text{Aut}(H)$ . Assume that  $\phi$  is  $\rho$ -liftable. Then  $\phi$  is either  $\rho$ -even, or  $\rho$ -odd, or  $\rho$ -neutral, and only one case occurs. Moreover, if  $H$  is semisimple and simply connected (or adjoint) and  $G = \text{SO}(2N, \mathbb{C})$ , then  $\rho$  is LFMO-special if and only if: (1)  $\rho$  has weight 1 and (2) each  $\phi \in \text{Aut}(H)$  is either  $\rho$ -even or not  $\rho$ -liftable.*

*Proof:*

Let  $B$  be a non-degenerate  $H$ -invariant symmetric / alternating bilinear form on  $V$ , the representation space of  $\rho$ , and  $\Omega = \text{O}(V, B)$ . Assume that  $\phi \in \text{Aut}(H)$  is  $\rho$ -liftable, and  $\phi$  lifts to  $T \in \text{GL}(V)$ . As  $\rho$  is stable and  $V$  is irreducible,  $B^T = CB$  for some  $C \in \mathbb{C}$  where  $B^T(u, v) = B(Tu, Tv)$  (Schur's lemma, Lemma 2.6). Thus  $\phi$  lifts to  $C^{-1}T \in \text{O}(V, B)$ . If  $B$  is alternating, then  $\Omega = G = \text{Sp}(2N, \mathbb{C})$  and  $\phi$  is  $\rho$ -even;

If  $B$  is symmetric and  $V$  is odd dimensional, then  $\Omega = G = \mathrm{SO}(2N + 1, \mathbb{C})$ , and  $\phi$  is  $\rho$ -neutral; If  $B$  is symmetric and  $V$  is even dimensional, then  $\Omega = \mathrm{O}(2N, \mathbb{C})$ , and  $\phi$  is either  $\rho$ -even or  $\rho$ -odd. Moreover, if  $\phi$  lifts to  $T, T' \in \Omega$ , then  $T^{-1}T'$  centralizes  $\rho(H)$ . As  $V$  is irreducible,  $T^{-1}T' \in \Omega$  is a scalar, and hence is  $\pm 1 \in G = \mathrm{SO}(2N, \mathbb{C})$ . Thus  $T \in G$  if and only if  $T' \in G$ . So  $\phi$  can't be  $\rho$ -even or  $\rho$ -odd at the same time.

Next assertion: Assume now that  $\rho$  has weight 1. Since  $\rho$  is assumed to be stable, and hence essential, by Theorem C, it suffices for us to show that the condition (3): If an automorphism  $\phi'$  of  $\rho(H)$  lifts to  $T \in \Omega$  then  $T \subset G$  is equivalent to the following:

Condition (x): all  $\rho$ -liftable automorphisms of  $H$  are  $\rho$ -even.

First (3) implies (x). Assume that  $\phi \in \mathrm{Aut}(H)$  is  $\rho$ -liftable, and lifts to  $T \in \Omega$ . Then the conjugation  $c_T$  induces an automorphism of  $\rho(H)$ , and thus by (3),  $c_T$  must be inner, and hence  $\phi$  is  $\rho$ -even.

Conversely, (x) implies (3). Assume that  $\phi' \in \mathrm{Aut}(\rho(H))$  lifts to an automorphism  $L$  on  $G$ . Then by Lemma 3.5 and the assumption that  $H$  is simply connected or adjoint, there is a  $\phi \in \mathrm{Aut}(H)$  such that  $\rho \circ \phi = \phi' \circ \rho$  (also see the diagram in the proof of Lemma 3.5). Thus  $\phi$  is  $\rho$ -liftable, and by (x), is  $\rho$ -even. and hence  $L$  must be inner, and hence a conjugation on  $G$ .

□

**Lemma 3.7.** *Let  $H$  be a semisimple, simply connected or adjoint Lie group, and  $H = H_1 \times H_2 \times \dots \times H_r$  with  $H_i$  simple. Then  $\mathrm{Aut}(H)$  is generate by the following families:*

(Type 1: Decomposable):  $\phi_1 \times \phi_2 \times \dots \times \phi_r$  where  $\phi_i \in \mathrm{Aut}(H_i)$ .

(Type 2: Swapping):

$$\begin{aligned} T_\lambda(g_1, g_2, \dots, g_i, \dots, g_j, \dots, g_r) \\ = (g_1, g_2, \dots, \lambda^{-1}(g_j), \dots, \lambda(g_i), \dots, g_r) \end{aligned}$$

where  $\lambda : H_i \xrightarrow{\cong} H_j$  is an isomorphism.

*Proof:* Let  $T \in \mathrm{Aut}(G)$ . Then  $T(H_i)$  is a normal simple subgroup of  $H$ . As  $H$  is simply connected /adjoint,  $T(H_i) = H_j$  for some  $j$ . Put  $\sigma \in S_n$  such that  $T(H_i) = H_{\sigma(i)}$ . Then  $H_i \cong H_{\sigma(i)}$ . Thus for each cycle  $C$  in  $\sigma$  and  $H_i$  are isomorphic all indices  $i$  occurred  $C$ . Thus, there is a product of Type 2 automorphisms  $T'$  with  $T'(H_i) = T(H_i)$  for all  $i$ . Note that  $T'^{-1} \circ T$  is of Type 1.

□

**Theorem 3.8.** *Let  $H$  be a semisimple, simply connected or adjoint Lie group, and  $H = H_1 \times H_2 \times \dots \times H_r$  with  $H_i$  simple. Let  $\rho = \bigotimes_i \rho_i$  where  $\rho_i$  is a finite dimensional irreducible complex representation of  $H_i$ . Then  $\mathrm{Aut}(H, \rho)$ , the set of  $\rho$ -liftable automorphisms of  $H$ , is generated by the following families*

(Type 1L: Decomposable):  $\phi_1 \times \phi_2 \times \dots \times \phi_r$  where  $\phi_i \in \mathrm{Aut}(H_i, \rho_i)$ .

(Type 2L: Swapping):

$$\begin{aligned} T_\lambda(g_1, g_2, \dots, g_i, \dots, g_j, \dots, g_r) \\ = (g_1, g_2, \dots, \lambda^{-1}(g_j), \dots, \lambda(g_i), \dots, g_r) \end{aligned}$$

where  $\lambda : H_i \xrightarrow{\cong} H_j$  is any isomorphism, and  $\rho_i, \rho_j \circ \lambda$  are equivalent.

**Lemma 3.9.** *Let  $H$  and  $H'$  be two semisimple Lie groups, which are both simply connected or adjoint,  $\rho, \rho'$  finite dimensional complex representations of  $H$  and  $H'$  respectively. Then the following are equivalent: (a)  $\rho$  and  $\rho'$  are quasi-equivalent. (b) There is an isomorphism  $\lambda : H \rightarrow H'$  such that  $\rho$  and  $\rho' \circ \lambda$  are equivalent.*

*Proof:* It is obvious that (b) implies (a). Now assume (a). Then  $H$  and  $H'$  are isogenous and we have the following diagram

$$\begin{array}{ccccc} H'' & \xrightarrow{\pi} & H & \xrightarrow{\rho} & \mathrm{GL}(V) \\ & \searrow \pi' & & & \downarrow \cong \\ & & H' & \xrightarrow{\rho'} & \mathrm{GL}(V) \end{array}$$

for some finite central isogenies  $\pi$  and  $\pi'$ . If both  $H$  and  $H'$  are simply connected, then  $\pi$  and  $\pi'$  are isomorphisms. If both  $H$  and  $H'$  are adjoint, then by factoring by the center, can choose  $H''$  also adjoint, so that  $\pi$  and  $\pi'$  are isomorphisms.

□

**Proposition 3.10.** *Let  $H, H'$  be two semisimple complex Lie groups and  $\rho, \rho'$  two finite dimensional irreducible complex representations of  $H, H'$  respectively. Let  $\phi \in \mathrm{Aut}(H)$  and  $\phi' \in \mathrm{Aut}(H')$ .*

(1)  $\phi \times \phi'$  is  $\rho \otimes \rho'$ -liftable if and only if  $\phi$  is  $\rho$ -liftable and  $\phi'$  is  $\rho'$ -liftable.

(2) Assume that  $\lambda : H \rightarrow H'$  is an isomorphism. Then  $T_\lambda : H \times H' \rightarrow H \times H'$  defined as

$$T_\lambda(h, h') = (\lambda^{-1}(h'), \lambda(h))$$

is  $\rho \otimes \rho'$ -liftable if and only if  $\rho, \rho' \circ \lambda$  are equivalent. In particular, if  $\rho$  and  $\rho'$  are quasi-equivalent, and both  $H$  and  $H'$  are simply connected or adjoint, then there is an isomorphism  $\lambda : H \rightarrow H'$  such that  $\rho = \rho' \circ \lambda$  and  $T_\lambda$  is  $\rho \otimes \rho'$ -liftable.

Now in addition, assume that  $\rho$  and  $\rho'$  are self-dual. We have:

(3) Assuming that  $\phi'$  is  $\rho'$ -neutral. Then  $\phi \times \phi'$  is  $\rho \otimes \rho'$ -neutral (resp.  $\rho \otimes \rho'$ -even, /  $\rho \otimes \rho'$ -odd) if and only if  $\phi$  is  $\rho$ -neutral (resp.  $\rho$ -even,  $\rho$ -odd).

(4) If  $\phi$  are  $\rho$ -odd or  $\rho$ -even and  $\phi'$  are  $\rho'$ -odd or  $\rho'$ -even, then  $\phi \times \phi'$  are  $\rho \otimes \rho'$ -even.

(5) Same assumption as in (2). Then  $T_\lambda(h, h')$  is  $\rho \otimes \rho'$ -neutral if and only if  $\rho$  is odd dimensional,  $\rho \otimes \rho'$ -even if and only if  $\dim(\rho) \equiv 0 \pmod{4}$  and  $\rho \otimes \rho'$ -odd if and only if  $\dim(\rho) \equiv 2 \pmod{4}$ .



*Proof:*

(1)  $\phi \times \phi'$  is  $\rho \otimes \rho'$ -liftable if and only if  $(\rho \otimes \rho') \circ (\phi \times \phi') = (\rho \circ \phi) \otimes (\rho' \circ \phi')$  and  $\rho \otimes \rho'$  are equivalent, if and only if  $\rho \circ \phi$  and  $\rho$  are equivalent and  $\rho' \circ \phi'$  and  $\rho'$  are equivalent, if and only if  $\phi$  is  $\rho$ -liftable and  $\phi'$  is  $\rho'$ -liftable.

(2) First, assume that  $T_\lambda$  is  $\rho$ -liftable. Then  $(\rho \otimes \rho') \times T_\lambda = (\rho' \circ \lambda^{-1}) \otimes (\rho \circ \lambda)$  and  $\rho \otimes \rho'$  are equivalent, and then  $\rho$  and  $\rho' \circ \lambda$  are then equivalent.

Next, assume that  $\rho$  and  $\rho' \circ \lambda$  are equivalent. Then  $\rho \circ \lambda^{-1}$  and  $\rho'$  are also equivalent, and hence  $(\rho \otimes \rho') \times T_\lambda = (\rho' \circ \lambda^{-1}) \otimes (\rho \circ \lambda)$  and  $\rho \otimes \rho'$  are equivalent. Hence  $T_\lambda$  is  $\rho \times \rho'$ -liftable.

Finally, if  $\rho$  and  $\rho'$  are quasi-equivalent and  $H$  and  $H'$  are both simply connected or both adjoint, there is a  $\lambda : H \xrightarrow{\cong} H'$  such that  $\rho = \rho' \circ \lambda$ , and hence  $T_\lambda$  is  $\rho \otimes \rho'$ -liftable.

(3) & (4) Let  $V, V'$  be the representation spaces of  $\rho, \rho'$  respectively, and  $B$  the non-degenerate  $H$ -invariant bilinear form of  $V$  and  $B'$  the non-degenerate  $H'$ -invariant bilinear form of  $V'$ . Let  $\Omega = O(V, B)$  and  $\Omega' = O(V', B')$ . Assume that  $\phi, \phi'$  lift to  $A \in \Omega$ ,  $A' \in \Omega'$  (see Proposition 3.6).

Then  $\phi \times \phi'$  lift to  $A \otimes A' \in \tilde{\Omega} = O(V \otimes V', B \otimes B')$ . Note that

$$\det(A \otimes A') = \det(A)^{\dim(\rho')} \det(A')^{\dim(\rho)}$$

while  $\det(A)$  and  $\det(A')$  are  $\pm 1$ .

Now we divide in cases.

(Case 3-1): Both  $\rho$  and  $\rho'$  are odd dimensional. In this case,  $\dim(\rho \otimes \rho')$  is odd, and hence if  $\phi$  is  $\rho$ -neutral, then  $\phi \times \phi'$  is  $\rho \otimes \rho'$ -neutral.

(Case 3-2):  $\rho'$  is odd dimensional and  $\rho$  is symplectic. In this case,  $\rho \otimes \rho'$  is also symplectic, and hence  $\phi$  is  $\rho$ -even and  $\phi \times \phi'$  is  $\rho \otimes \rho'$ -even.

(Case 3-3):  $\rho'$  is odd dimensional and  $\rho$  is even dimensional and orthogonal. In this case,  $\rho \otimes \rho'$  is also even dimensional orthogonal. Moreover,  $\det(A \otimes A') = \det(A)$ . as  $\phi \times \phi'$  lifts to  $A \otimes A' \in \tilde{\Omega} = O(V \otimes V', B \otimes B')$ . Hence if  $\phi$  is  $\rho$ -even (resp.  $\rho$ -odd), then  $A \in SO(V, B)$ ,  $\det(A)$  is 1 (resp.  $-1$ ),  $\det(A \otimes A')$  is 1 (resp.  $-1$ ),  $A \otimes A' \in SO(V \otimes V', B \otimes B')$ , and thus  $\phi \times \phi'$  is  $\rho \otimes \rho'$ -even.

(Case 4): All  $\rho$  and  $\rho'$  are even dimensional. In this case,  $\rho \otimes \rho'$  is also even dimensional. Moreover,  $\phi \times \phi'$  lifts to  $A \otimes A' \in \tilde{\Omega} = O(V \otimes V', B \otimes B')$  and  $\det(A \otimes A') = 1$ . Hence  $A \otimes A' \in SO(V \otimes V', B \otimes B')$ , and thus  $\phi \times \phi'$  is  $\rho \otimes \rho'$ -even.

(5) Now by assumptions in (2),  $\rho$  and  $\rho' \circ \lambda$  are conjugate, where  $\lambda : H \xrightarrow{\cong} H'$ . Hence we have an  $H$ -equivariant isometry  $E$  from  $V$  to  $V'$  (Proposition 2.2).

Then by (2),  $T_\lambda$  is  $\rho \otimes \rho'$  liftable, and it lifts to some isometry  $A_\lambda : V \otimes V' \rightarrow V \otimes V'$ , where  $A_\lambda(v \otimes v') = (E^{-1}(v') \otimes E(v))$  for each  $v \in V, v' \in V'$ . As  $\det(A_\lambda) = (-1)^{n(n-1)}$  where  $n = \dim(\rho)$ , then it is  $-1$  if  $n \equiv 2 \pmod{4}$ , and  $1$  if  $n \equiv 0 \pmod{4}$ . (Recall that the signature of  $\sigma : X \times X \rightarrow X \rightarrow X$  that sends  $(x, y)$  to  $(y, x)$  is  $(-1)^{|X|(|X|-1)}$ .) Thus,  $T_\lambda$  is  $\rho \otimes \rho'$ -neutral if  $\dim(\rho)$  is odd,  $T_\lambda$  is  $\rho \otimes \rho'$ -odd if  $\dim(\rho) \equiv 2 \pmod{4}$ , and  $T_\lambda$  is  $\rho \otimes \rho'$ -even if  $\dim(\rho) \equiv 0 \pmod{4}$ .

□

*Proof of Theorem 3.8:* We proceed in steps.

(Step 1): In two families in Lemma 3.7, the  $\rho$ -liftable ones are exactly in Type 1L and Type 2L.

First, Let  $\phi = \phi_1 \times \dots \times \phi_r \in \text{Aut}(H)$  of Type 1. Then by Proposition 3.10 (1),  $\phi$  is  $\rho$ -liftable if and only if  $\phi_i$  is  $\rho_i$ -liftable. So Type 1 &  $\rho$ -liftable  $\Leftrightarrow$  Type 1L.

Next, like the proof of Proposition 3.10 (2), let  $\phi = T_\lambda$  of Type 2 as in Lemma 3.7, and  $\rho' = \rho \circ \phi$ . Then  $\rho' = \bigotimes_i \rho'_i$  where  $\rho'_i = \rho_j \circ \lambda$ ,  $\rho'_j = \rho_i \circ \lambda^{-1}$ , and  $\rho'_k = \rho_k$  for  $k \neq i, j$ .  $\phi$  is  $\rho$ -liftable, if and only if  $\rho$  and  $\rho'$  are equivalent, if and only if  $\rho_k$  and  $\rho'_k$  are equivalent, if and only if  $\rho_i$  and  $\rho_j \circ \lambda$ ,  $\rho_j$  and  $\rho_i \circ \lambda^{-1}$  are equivalent, if and only if  $\rho_i$  and  $\rho_j \circ \lambda$  are equivalent. Hence Type 2 &  $\rho$ -liftable  $\Leftrightarrow$  Type 2L.

(Step 2): General case. For  $\phi \in \text{Aut}(H)$ , we have  $\sigma \in S_n$  such that  $\phi(H_i) = H_{\sigma(i)}$  (see Lemma 3.7 and its proof). Thus by the same argument as last step,  $\rho_i$  and  $\rho_j \circ \phi$  are equivalent, and hence  $H_i$  and  $H_j$  are quasi-equivalent. Thus for all  $i$  occurred in a cycle  $C$  of  $\sigma$ ,  $H_i$  are isomorphic since they are isogenous and both simply connected or both adjoint. Write  $\sigma = t_1 t_2 \dots t_l$  for transposition  $t_i$  where  $i_1, i_2$ , all indices of each  $t_i = (i_1 i_2)$  occur in the same cycle in  $\sigma$ . Then from Step 1, there is a  $\phi_i = T_{\lambda_i}$  of Type 2L, where  $\lambda_i : H_{i_1} \rightarrow H_{i_2}$  an isomorphism. Let  $T = \phi_l \circ \phi_{l-1} \circ \dots \circ \phi_1$ . Then  $T$  is  $\rho$ -liftable and  $T(H_i) = \phi(H_i)$  for all  $i$ . Thus  $T^{-1}\phi$  is of Type 1, and hence Type 1L. Hence the theorem.

□

**Proposition 3.11.** *All notations and assumptions as in Theorem 3.8. Assume that  $\rho$  are self-dual. Then*

(Type 1L): *Let  $\phi = \phi_1 \times \dots \times \phi_r$  and assume that it is  $\rho$ -liftable. Then  $\phi$  is  $\rho$ -neutral if and only if  $\rho_i$  is odd dimensional for all  $i$ .  $\phi$  is  $\rho$ -odd if and only if for exactly one  $i$ ,  $\phi_i$  are  $\rho_i$ -odd, and for all other  $j$ ,  $\rho_j$  are odd dimensional.  $\phi$  is  $\rho$ -even if and only one of the following happens: Either for some  $i$ ,  $\phi_i$  is  $\rho_i$ -even, or for at least two  $i \neq j$ ,  $\rho_i$  and  $\rho_j$  are even dimensional.*

(Type 2L): *Let*

$$\begin{aligned} T_\lambda(g_1, g_2, \dots, g_i, \dots, g_j, \dots, g_r) \\ = (g_1, g_2, \dots, \lambda^{-1}(g_j), \dots, \lambda(g_i), \dots, g_r) \end{aligned}$$

where  $\lambda : H_i \xrightarrow{\cong} H_j$  is any isomorphism. Assume that  $\rho_i, \rho_j \circ \lambda$  are equivalent. Then  $T_\lambda$  is  $\rho$ -neutral if and only if for all  $k$ ,  $\rho_k$  are odd dimensional,  $T_\lambda$  is  $\rho$ -odd if and only if  $\dim(\rho_i) \equiv 2 \pmod{4}$ , and for all  $k \neq i, j$ ,  $\rho_k$  is odd dimensional,  $T_\lambda$  is  $\rho$ -even if and only if  $\dim(\rho_i) \equiv 0 \pmod{4}$ , or for some  $k \neq i, j$ ,  $\rho_k$  is even dimensional.

*Proof:*

(1) First Type 1L. This follows from Proposition 3.10 (1), (3) and (4).

(2) Now work for Type 2L. If  $r = 2$ , then the assertions follow from Proposition 3.10 (2), (5). For general  $r$ , without loss of generality may assume that  $i = 1, j = 2$ . Let  $H' = H_1 \times H_2$ ,  $H'' = H_3 \times \dots \times H_r$ ,  $\rho' = \rho_1 \otimes \rho_2$ ,  $H'' = \rho_3 \otimes \dots \otimes \rho_r$  and  $\phi' = T_\lambda|_{H'}$ , and  $\phi = T_\lambda = \phi' \times 1$  on  $H = H' \times H''$ . Then  $\phi'$  is  $\rho'$ -liftable. Now Work in case by case carefully.  $\phi$  is  $\rho$ -neutral if and only if all  $\rho_k$  are odd

dimensional ((1) in this proof). If  $\rho''$  is odd dimensional, then  $\phi$  is  $\rho$ -odd (resp.  $\rho$ -even) if and only if  $\phi'$  is  $\rho'$ -odd (resp.  $\rho'$ -even) ((1) in this proof), if and only if  $\dim(\rho_1) \equiv 2 \pmod{4}$  (resp.  $\dim(\rho_1) \equiv 0 \pmod{4}$ ). If  $\rho''$  is even dimensional, then  $1|_{H''}$  is always  $\rho''$ -even,  $\phi$  is then  $\rho$ -even.

□

**Theorem 3.12.** *Let  $H$  be a semisimple, simply connected or adjoint Lie group, and  $H = H_1 \times H_2 \times \dots \times H_r$  with  $H_i$  simple. Let  $\rho = \bigotimes_i \rho_i$  where  $\rho_i$  is a finite dimensional irreducible complex representation of  $H_i$ .*

*Assume that  $\rho$  is even dimensional orthogonal. Then all  $\rho$ -liftable automorphisms of  $H$  are  $\rho$ -even if and only if one of the following case happens.*

Case (1): *Exactly one  $\rho_i$  is even dimensional. In this case all  $\rho_i$ -liftable automorphisms of  $H_i$  are  $\rho_i$ -even.*

Case (2): *Exactly two, say  $\rho_i$  and  $\rho_j$ , are even dimensional, and moreover, either  $\rho_i$  and  $\rho_j$  are not quasi-equivalent, or  $\dim(\rho_i) \equiv 0 \pmod{4}$ .*

Case (3): *At least three  $\rho_i$ s are even dimensional.*

**Corollary 3.13.** *All notations and assumptions are same as in Theorem 3.12. Then  $\rho$  is LFMO-special, if and only if each  $\rho_i$  has weight 1, and one of the following holds:*

Case (1): *Exactly one  $\rho_i$  is even dimensional. In this case  $\rho_i$  is LFMO-special.*

Case (2): *Same as Case (2) in Theorem 3.12.*

Case (3): *Same as Case (2) in Theorem 3.12.*

*Proof of Corollary of Corollary 3.13 using Theorem 3.12:* Note that if Case (1) occurs, then  $\rho_i$  must be orthogonal since all other  $\rho_j$  are odd dimensional orthogonal, and  $\rho$  is also orthogonal. Now this corollary follows directly from Theorem 3.12, Proposition 3.6 and Lemma 3.7.

□

*Proof of theorem 3.12:*

From Theorem 3.8, it suffices for us to check the  $\rho$ -liftable automorphisms of  $H$  in Type 1L and Type 2L.

Step 1: Only if part. If exactly one  $\rho_i$  is even dimensional, and some  $\phi_i \in \text{Aut}(H_i)$  is  $\rho_i$ -odd, then  $\phi' = 1 \times \dots \times \rho_i \times \dots \times 1$  is also  $\rho$  odd (Proposition 3.11, Type 1L); If exactly two, say  $\rho_i$  and  $\rho_j$  are even dimensional, and  $\rho_i$  and  $\rho_j$  are quasi-equivalent, then there is a  $\lambda : H_i \xrightarrow{\cong} H_j$  such that  $T_\lambda$  is  $\rho$ -odd. So the “only if” part follows.

Step 2: Assume that all conditions in three cases hold. Want to prove that all  $\rho$ -liftable automorphisms of  $H$  in Type 1L and Type 2L are  $\rho$ -even.

Case (1) + Type 1L: Consider  $\phi = \phi_1 \times \dots \times \phi_r \in \text{Aut}(H)$  in as in Type 1L of Proposition 3.11. As  $\phi_i$  is  $\rho_i$ -even from the assumption of Case (1), and all other  $\phi_j$  are  $\rho_j$ -neutral as  $\rho_j$  are odd dimensional,  $\phi$  is then  $\rho$ -even (Proposition 3.11, Type 1L).

Case (1) + Type 2L: Consider  $T_\lambda$  as in Type 2L of Proposition 3.11 where  $\lambda : H_i \xrightarrow{\cong} H_j$ . Then  $\rho_i$  and  $\rho_j$  are odd dimensional, and thus for some unique  $k \neq i, j$ ,  $\rho_k$  is even dimensional. Then  $T_\lambda$  is  $\rho$ -even. (Proposition 3.11, Type 2L)

Case (2) & (3) + Type 1L: Consider  $\phi = \phi_1 \times \dots \times \phi_r \in \text{Aut}(H)$  in as in Type 1L of Proposition 3.11. As  $\phi_i$  is  $\rho_i$ -even and  $\rho_j$  is  $\rho_j$ -even for  $j \neq i$  from the assumption of Case (2) and (3),  $\phi$  is then  $\rho$ -even (Proposition 3.11, Type 1L).

Case (2) + Type 2L: Consider  $T_\lambda$  as in Type 2L of Proposition 3.11. If  $\rho_i$  and  $\rho_j$  are even dimensional, then, as  $\dim(\rho_i) \equiv 0 \pmod{4}$ ,  $T_\lambda$  is  $\rho$ -even (Proposition 3.11, Type 2L). If  $\rho_i$  and  $\rho_j$  are odd dimensional, then there must be two indices  $k, l$  with  $\rho_k$  and  $\rho_l$  are even dimensional, and thus  $T_\lambda$  is also  $\rho$ -even (Proposition 3.11, Type 2L).

Case (3) + Type 2L: Consider  $T_\lambda$  as in Type 2L of Proposition 3.11. No matter whether  $\rho_i$  and  $\rho_j$  are even dimensional, there must be a  $k \neq i, j$  with  $\rho_k$  is even dimensional, and thus  $T_\lambda$  is also  $\rho$ -even (Proposition 3.11, Type 2L).

Hence all  $\rho$ -liftable automorphisms of  $H$  of Type 1L and Type 2L are  $\rho$ -even. By Theorem 3.8, all  $\rho$ -liftable automorphisms are  $\rho$ -even. Hence the theorem.  $\square$

**Proposition 3.14.** *Let  $H$  be a connected complex reductive group and  $\rho$  be a finite dimensional complex representation. If  $\rho(H)$  has trivial center, then  $\rho$  has weight 1. The converse is true if  $\rho$  is irreducible.*

*Proof:* The weight set of  $\rho$  forms a saturated set in  $\Lambda$  (Proposition 21.3 of [11]), the weight lattice of  $T$ . Then the first assertion for  $H$  semisimple follows from Lemma 13.4B of [11]. For reductive  $H$ , since  $\rho$  factors through its quotient which is semisimple of adjoint type, then the first assertion for general case follows also from the semisimple case.

For the converse, since  $\rho$  is irreducible,  $Z(H)$ , the center of  $H$  acts as scalars (Schur's Lemma, cf. Lemma 2.6), and hence we have the converse.  $\square$

Now we come to our main theorems.

*Proof of Theorem D:*

We also proceed in steps. Recall that  $H = H_1 \times \dots \times H_r$ , and  $\rho = \bigotimes_i \rho_i$  where  $\rho_i$  is a self-dual representation of  $H_i$ .

Step (1):  $H$  is simply connected (resp. adjoint). i.e.,  $T = 1$  and all  $H_i$  are simply connected (resp. adjoint). Note that the center of  $\rho(H)$  is trivial if and only if  $\rho$  has weight 1 (Proposition 3.14). Hence Theorem D is directly from Corollary 3.13.

Step (2): General semisimple  $H = H_1 \times \dots \times H_r$ .

Let  $\tilde{H} = \tilde{H}_1 \times \dots \times \tilde{H}_r$  where  $\tilde{H}_i$  is the simply connected cover of  $H_i$ , together with the covering maps  $\pi_{H_i} : \tilde{H}_i \rightarrow H_i$ , and  $\pi_H : \tilde{H} \rightarrow H$  compatible to  $\pi_{H_i}$ . Let  $\tilde{\rho}_i = \rho_i \circ \pi_{H_i}$  and  $\tilde{\rho} = \rho \circ \pi_H = \bigotimes_i \tilde{\rho}_i$ .

Then as  $\pi_H$  is a finite central isogenies, from the definition, we see easily that  $\rho$  is LFMO-special, if and only if  $\tilde{\rho}$  is LFMO-special. In fact, all the three conditions in the definitions are the same for  $\rho$  and  $\tilde{\rho}$ . Also, similarly the same for  $\rho_i$  and  $\tilde{\rho}_i$ .

Hence for each statement of the assertions in all three cases in Theorem D are equivalent for  $H$  and for  $\tilde{H}$ . (Note that by our definition,  $\rho_i$  and  $\tilde{\rho}_i$  are quasi-equivalent! So the assertion above works for Case (2).) Then finally, Theorem D for  $H$  is equivalent to Theorem D for  $\tilde{H}$ , which is already done in Step (1).

Step (3): General  $H = T \times H_0$  where  $H_0 = H_1 \times \dots \times H_r$  the derived group of  $H$ . Now Theorem D follows from Step (2) since  $\rho$  is LFMO-special if and only  $\rho|_{H_0}$  is so.

□

**Theorem 3.15.** *Let  $H$  be a simple group of adjoint type and  $\rho$  be the adjoint representation of  $H$ . Then  $\rho$  is LFMO-special if and only if  $H$  is of the type  $A_{4n}(n \geq 1), B_{2n}(n \geq 1), C_{2n}(n \geq 2), E_6, E_8, F_4, G_2$ .*

This is also Theorem A in [28]. See Table 3.1.

*Proof:* Let  $\mathfrak{h}$  be the Lie algebra of  $H$ ,  $\kappa$  the Killing form of  $\mathfrak{h}$ ,  $G = SO((\mathfrak{h}), \kappa)$ , and  $\Omega = O(\mathfrak{h}, \kappa)$ . Then  $\rho$  is orthogonal and its image in  $GL(\mathfrak{h})$  lies in  $G$ . Moreover,  $\rho$  definitely has weight 1. Thus,  $\rho$  is LFMO-special if and only if all  $\phi \in \text{Aut}(H)$  is  $\rho$ -even. In this case,  $\dim(\rho) = \dim(\mathfrak{h})$  is even, and equivalently,  $H$  has even rank. So we focus on adjoint simple groups  $H$  of type  $A_{2n}(n \geq 1), B_{2n}(n \geq 1), C_{2n}(n \geq 2), D_{2n}(n \geq 3), E_6, E_8, F_4, G_2$ .

Let  $T$  be a maximal torus of  $H$  with its Lie subalgebra  $\mathfrak{t} \in \mathfrak{h}$ ,  $\Phi$  the root set of  $\mathfrak{h}$  with respect to  $\mathfrak{t}$  and  $\Delta$  a simple basis. Fix  $\Sigma = (\mathfrak{t}, \{u_\alpha\}(\alpha \in \Delta))$  where  $u_\alpha$  an eigenvector in  $\mathfrak{h}$  of the root  $\alpha$ .

Then by basic Lie theory,

$$\text{Aut}(H) = \text{Int}(H) \rtimes \text{Aut}(H, \Sigma)$$

where  $\text{Aut}(H, \Sigma)$  is the set of  $\phi \in \text{Aut}(H)$  that fixes  $\Sigma$ . In particular, can choose an eigenbasis of  $T$ :  $(t_\alpha(\alpha \in \Delta), u_\beta(\beta \in \Phi))$  such that  $\text{Aut}(H, \Sigma)$  is a group of permutations of coordinates. In particular,  $\text{Aut}(H, \Sigma) \in \Omega$ .

Since each  $\phi \in \text{Aut}(H)$  is  $\rho$ -liftable, and lifts to  $\phi$  itself when identify  $\text{Aut}(H)$  with  $\text{Aut}(\mathfrak{h})$  via the adjoint representation  $\rho$ , by the discussion above, we have, to prove that all  $\phi$  are  $\rho$ -even, it suffices to prove that all  $\phi \in \text{Aut}(H, \Sigma)$  are even coordinate permutations.

For each  $\phi \in \text{Aut}(H, \Sigma)$ ,  $\phi$  permute the positive root set  $\Phi^+$  and the negative root set  $\Phi^- = -\Phi^+$  in exactly the same style. Hence the signature of  $\phi$  agrees with the signature of  $\phi$  on  $\Delta$ , or equivalently, the Dynkin diagram.

Hence for  $H$  of even rank,  $\rho$  is LFMO-special if and only if all automorphisms of  $H$  are  $\rho$ -even, if and only if all automorphisms in  $\text{Aut}(H, \Sigma)$  are even coordinate permutations, if and only if all automorphisms of the Dynkin diagram of  $H$  are even permutations. Now Table 3.1 will finally conclude our theorem.

Note that finally we get  $A_{4n}(n \geq 1), B_{2n}(n \geq 1), C_{2n}(n \geq 2), E_6, E_8, F_4, G_2$ .

□

*Proof of Corollary E:* From the facts below about the simple factor  $H_j$  which is adjoint, we see that the corollary follows directly from Theorem D and Theorem 3.15. First, adjoint representation of  $H_i$  has weight 1. Second, the adjoint

TABLE 3.1. Dynkin Diagrams of Even Rank

Type	Diagram	Generators of Automorphism Group	Signatures
$A_{2n}$ ( $n \geq 1$ )		$(1 \ 2n)(2 \ 2n-1) \cdots (n+1 \ n)$	$(-1)^n$ $\checkmark$ for $2 n$ $\times$ for $2 \nmid n$
$B_{2n}$ ( $n \geq 2$ )		1	$\checkmark$
$C_{2n}$ ( $n \geq 2$ )		1	$\checkmark$
$D_{2n}$ ( $n \geq 3$ )		$(2n-1 \ 2n)$ if $n \neq 4$ $(3 \ 4), (1 \ 3 \ 4)$ if $n = 4$	$-1$ if $n \neq 4$ $-1, 1$ if $n = 4$ $\times$ Always!
$E_6$		$(1 \ 6)(3 \ 5)$	$1 \ \checkmark$
$E_8$		1	$\checkmark$
$F_4$		1	$\checkmark$
$G_2$		1	$\checkmark$

representations of  $H_i$  and  $H_j$  are quasi-equivalent if and only if  $H_i$  and  $H_j$  are isomorphic. Third,  $\dim(\rho_j) = \dim(H_j)$  is even if and only if  $H_j$  has even rank.

□

#### 4. ODDS AND ENDS

##### 4.1. $E_8$ .

At the end of this notes, we quote the following theorem, which follows from A result of Seitz ([21], [22]).

**Theorem 4.1.** *Let  $H = \mathrm{PGL}(3, \mathbb{C})$  and  $G = E_8(\mathbb{C})$ . Then there exists an embedding  $\rho : H \hookrightarrow G$  such that  $m'(\rho; G) > 1$ .*

□

*Remark:* There exists a homomorphism  $i : \mathrm{Spin}(16, \mathbb{C}) \hookrightarrow E_8(\mathbb{C})$  with kernel  $\mathbb{Z}/2\mathbb{Z}$ . Let  $\rho'_1 : \mathrm{PGL}(3, \mathbb{C}) \rightarrow \mathrm{SL}(8, \mathbb{C})$  be the adjoint representation, and  $\rho' = \rho'_1 \oplus \rho'_1$ . Then  $\rho'$  is orthogonal type. Since  $Z(\mathrm{SL}(3, \mathbb{C})) \cong \mathbb{Z}/3\mathbb{Z}$ ,  $\rho'$  also lifts to

$\tilde{\rho}' : \mathrm{PGL}(3, \mathbb{C}) \rightarrow \mathrm{Spin}(16, \mathbb{C})$ . Then  $\rho = i \circ \tilde{\rho}'$ . From C,  $\rho'$  is LFMO-special and hence  $m'(\tilde{\rho}; \mathrm{Spin}(16, \mathbb{C})) = 2$  (Theorem 2.8, Theorem 2.14).

#### 4.2. Final Remark.

Since all arguments and statements are worked through complex linear algebraic groups, and we don't involve any special topology, then everything and finally all results work also for  $K$ , an algebraic closed field of characteristic 0, for example,  $\bar{\mathbb{Q}}_l$ . Moreover, when  $H$  and  $G$  are connected, we have also Lie algebra version of our concepts and results. In fact, some of well known results were first introduced as Lie algebra version (see [7], [8]). Moreover, we have also the compact real reductive group version. Our paper just focuses on the version of the algebraic group over  $\mathbb{C}$ .

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